Equivariant Dynamical Systems

by

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Introduction

In this paper we study certain classes of maps invariant by a compact Lie group action*. The work is divided into three parts. Part 1 is chiefly concerned with spaces of C^r equivariant maps $(r>0)$ & the main result is to show that the space of equivariant C^r sections of a G-fiber bundle, compact base & paracompact fiber, '$C^r_G(E)$', may be given the structure of a C^∞ Banach submanifold of $C^r(E)$.

In Part 2, we develop definitions of genericity for equivariant vector fields & diffeomorphisms. After some stable manifold theory, we prove a Kupka-Smale density theorem for equivariant vector fields.

In Part 3, we take a brief look at 'G-Morse-Smale' systems & 'G-Anosov maps', proving existance of the former & structural stability of the latter.

We will now describe the contents of each section in more detail. For convenience, we let '1.n' denote section n, Part 1 & 'A.n' denote Appendix n.

1.1 We give formal definitions & proofs for 'G-Banach spaces'; i.e. a G-Banach space is a Banach space, with a linear G-action & G-invariant norm.

1.2 We look briefly at $H^r_G(E)$, equivariant L^2_r sections of a G-fiber bundle E, & show how $H^r_G(E)$ may be easily shown to be a C^∞ submanifold of $H^r(E)$ (I am grateful to Prof. J. Eells for pointing this out to me).

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*Most of the work is valid for 'Proper G-Spaces', see Palais, Ann. Math. 73,2 (1961) 295-323.

1.3,1.4 & 1.5 are directed towards proving the main result of Part1,that $C^r_G(E)$ may be considered as a C^∞ Banach submanifold of $C^r(E)$.The proof of this result is a straightforward generalisation of the proof for G=id in Palais 1.

2.1 & A.1 consist of a brief résumé of the theory of differentiable actions of compact Lie groups needed in the sequel.

2.2 We define a rather specialised form of transversality for equivariant embeddings & prove an isotopy theorem.The only thing of interest here is the weakness of the transversality condition giving the isotopy result.

2.3 We prove a local Lemma which is fundamental in later applications of transversality theory.Using this Lemma we take a brief look at $J_G(M,N)$.

2.4 Using the Lemma of 2.3 we prove a density theorem about equivariant sections transversal to a compact submanifold of a G-fiber bundle.The proof is done in the spirit of Abraham 1.

2.5 We define '1-genericity' for fixed sets & singular sets of equivariant diffeomorphisms & vector fields. Essentially a fixed set is 1-generic if it is normally hyperbolic,see P-H-S 1.We show that given a fixed set of an equivariant C^r diffeomorphism f,we may C^r perturb f to f' s.t. f' s.t. G(x) becomes a 1-generic set for f',similarly for vector fields.

2.6 We show that the subset of equivariant diffeomorpisms all of whose fixed sets are 1-generic is an open & dense subset of the set of all equivariant diffeomorphisms.Similarly for diffeomorphisms.We show that we may insist that such fixed sets are generically stable,i.e. do not perturb into non-compact orbits.

2.7 We define the stabiliser group of a closed orbit & note that that there are two distinct types of closed orbit. We take a look at equivariant flows on Homogeneous spaces proving some elementary results which are of use later on.

2.8 We define 2-genericity for closed orbits,essentially by requiring normal hyperbolicity.One featüre here is that with the particular definition of genericity choosen,closed orbits are stable under perturbation.

2.9 & A.4 & A.3. In section 9 we develop some Floquet theory for closed orbits of equivariant vector fields. Essentially we pull back the flow in a tubular nbd. of the closed orbit to a universal cover of this nbd. & at the same time pulling back (part of) the G-action.Then we strighten out the flow in this cover.By this means we are able to do perturbation theory for closed orbits of equivariant vector fields.We are,however faced with differentiability problems for one type of closed orbit.In Appendix 4, we strengthen our results as far as possible for one type of orbit,in Appendix 3 we prove an easy perturbation theorem for the other type of orbit,with no loss of differentiability.

2.10 Using the results of 2.8 we prove some perturbation theorems about nbds of a closed orbit which we need in 2.13.

2.11 We develop some results on stable manifold theory for equivariant diffeomorphisms & vector fields, leaning heavily on P-H-S 1. We prove a parametrisation theorem for global stable manifolds, which we use in 2.15.

2.12 Again, using P-H-S 1, we prove a version of Hartman's theorem for equivariant maps.

2.13 We show that 2-genericity is a generic property. The argument here is essentially that of Peixoto 1, with additions.

2.14 We define G-transversality for stable & unstable manifolds, essentially by requiring transversality for each orbit type. This definition is similar to one in Wasserman 1, though more general.

2.15 We show that insisting that stable & unstable manifolds are G-transversal still allows us to prove genericity. The argument here is based on that in Abraham 1 & essentially consists of a little perturbation theory + a Baire category argument. We also show that the results generalise to non-compact manifolds without difficulty.

3.1, 3.2 & 3.3. Here we define a 'G-Morse-Smale system' & consider various definitions of structural stability & topology which make it reasonable to ask whether such systems are structurally stable. We show that a 'G-Morse-Smale system' exists on every compact G-manifold.

Finally we define a G-Anosov map & prove that such maps are structurally stable.

A.2 We define & prove results about G vector bundle approximations.

Acknowledgements

The work in this Thesis was done in the year 1969-1970 under the supervision of Prof. J.Eells. I would like to thank him for his never failing encouragement & stimulating conversation,without which the Thesis would not have attained its final form.

My supervisor for the year 1968-1969 was Prof. E.C.Zeeman & I am most grateful for his helpful supervision.

During the period 1968-1970 I was receiving an S.R.C. post-graduate grant.

Corrections

p. 5, 2nd line from bottom should read: If we define $L_r^2(E)=H^r=$ the completion of $\{s\varepsilon L^2(E): s$ is C^r, with i.p. induced from $L^2(J^r(E))\}$ it is well known that H^r is a Hilbert space.

p. 29, 8th line from top: The proof is some what incomplete from here on. We regard 'A' as a local section at (x,y) for L(TM,TN) and construct a nbd. of A in C^o-Topology, with the required properties using the continuity of the evaluation map and compactness properties.

p. 35, Lemma 2: The proof is unnecessarily complicated; replace K by S_1.

p. 58. The first part of Proposition 13 is false, as the last line of the proof is incorrect. The statement should be : '$M_i^x \times M_i^x$ is a submanifold of $(M \times M)_i^x$ of codimension 0'. The proof follows from the correct second paragraph of the proof of Proposition 13 given on page 58.

p. 59, 11th line from bottom: For 'codimension' read 'dimension'. Similarly for botton line of page 61.

p. 77, 9th line from bottom should read: R(G/H)=rank(N(H)/H).

p. 132, 2nd line from bottom should read: $W^s(x)=$ $\{z\varepsilon M: F_t^n z \longrightarrow F_t^n x$ as $n \longrightarrow oo\}, x\varepsilon G(q)$.

p. 182. Definition 39: Perhaps it ought to be stated that G-structural stability defines an equivalence relation. In fact it is easy to check this.

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PART 2 EQUIVARIANT VECTOR FIELDS

PART 3 MISCELLANEOUS RESULTS ON EQUIVARIANT VECTOR FIELDS

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Notations

$M, N, \ldots$ will denote differentiable manifolds, without boundary. Unless stated to the contrary, we always suppose them to be of class C^∞.

$\mathfrak{C}^r(M,N)$ will denote the set of C^r maps from M to N.

G will always denote a compact Lie group.

(M,G) will denote a C^∞ manifold, together with a compact Lie group of transformations G acting differentiably on it. Such manifolds will be termed G-manifolds &, in general, we will denote them by M, rather than by (G,M).

If M & N are G-manifolds & $f \in \mathfrak{C}^r(M,N)$, then we say f is equivariant if $fg = gf$ $(g \in G)$. We denote the set of C^r equivariant maps from M to N by $\mathfrak{C}^r_G(M,N)$.

$VB(M)$ will denote the category of C^∞ vector bundles over M.

$FB(M)$ will denote the category of C^∞ fiber bundles over M. (See, for example, Palais 1 for more details).

$VB^r(M)$ $(FB^r(M))$ will denote the category of C^r vector bundles (fiber bundles) over M, where M is of class C^r.

We define a G-vector bundle over a G-manifold M, to be a vector bundle $\phi: E \longrightarrow M$, belonging to $VB(M)$ s.t. E is a G-manifold, ϕ is equivariant &, for each $g \in G$, $g: E_x \longmapsto E_{gx}$ is a vector space homomorphism.

$GVB(M)$ will denote the category of C^∞ G vector bundles over M.

Similarly:

$GFB(M)$ will denote the category of C^∞ G fiber bundles over M.

$C^r(E)$ will denote the set of C^r sections of an element of $VB(M)$ ($FB(M)$).

$C^r_G(E)$ will denote the set of equivariant C^r sections of an element of $GVB(M)$ ($GFB(M)$).

Associated to a compact Lie group G, we have a unique normalised Haar measure, which we will always denote by 'dg'.

'$\pitchfork$' will denote transversal intersection (Abraham 1, page 45).

If $H\subset G$ as a subgroup, '$N(H)$' will denote the normaliser of H in G.

Tf will generally denote the differential of a map f, though Df (& df) will also be used on occasions.

If A is a subset of M, $\bar{A}$ & $\mathring{A}$ will respectively denote the closure & interior of A.

When we refer to 'G-normal bundles' we mean normal bundles of a G-invariant submanifold, constructed using an equivariant Riemannian metric, giving a G-vector bundle. See Kozul 1 or Borel 1 (Palais' lecture) or Wasserman 1.

'$\forall$' & '$\exists$' will respectively denote universal & existential quantification.

Part 1: Manifolds of Equivariant Maps.

1. G-Banach Spaces & Banach Spaces of Sections

Let $(E,\| \|)$ denote a Banach space E, together with norm $\| \|$. Suppose G acts as a group of (continuous) linear transformations on E.

Lemma 1

With the above notation, there is a norm, $\| \|_G$ on E, equivalent to $\| \|$, such that $\| \|_G$ is G-invariant, i.e.:

$$\|gx\|_G = \|x\|_G \quad (x \in E), (g \in G).$$

Proof

Define $\|x\|_G = \int_G \|gx\| dg$.

Clearly $\| \|_G$ is a norm on E, we must show that it is equivalent to $\| \|$.

Let $g \in G$, then $g: E \longrightarrow E$ is a Banach space isomorphism & $\exists \lambda_g, \mu_g$, strictly positive numbers s.t.:

$$\lambda_g \|x\| \leqslant \|gx\| \leqslant \mu_g \|x\|.$$

Let $\lambda = \inf_{g \in G} \lambda_g$, $\mu = \sup_{g \in G} \mu_g$. Then $\infty > \lambda, \mu > 0$, since G is compact.

Thus:

$$\lambda \|x\| \leqslant \int_G \|gx\| dg \leqslant \mu \|x\|.$$

i.e. : $\lambda \|x\| \leqslant \|x\|_G \leqslant \mu \|x\|$.

q.e.d.

Definition 1

Say $(E, \| \|, G)$ is a G-Banach space if (E,G) is a G-Banach manifold & $\| \|$ is G-invariant; G acts linearly on E.

Lemma 1 implies that if $(E, \| \|)$ is a Banach Space & (E,G)

a G-Banach manifold, then we may assume that $(E,\|\ \|,G)$ is a G-Banach space; G, of course, is assumed to act linearly.

Let B_G denote the category of G-Banach spaces & continuous equivariant Banach space maps.

If $E \in VB(M)$ & M is compact, then we may define the C^r topology on $C^r(E)$ & this topology is defined by a C^r norm $\|\ \|_r$. $C^r(E)$ is a Banach space w.r.t. $\|\ \|_r$ (for details, see, for example, Abraham 1).

Now suppose $E \in GVB(M)$, then we have a natural G-action on $C^r(E)$ given by $s \longmapsto g \cdot s \cdot g^{-1}$. This action is clearly linear: it is also continuous & therefore C^∞. To see this, we note that the C^r topology on $C^r(E)$ may be induced from a C^0 topology on $C^0(J^r(E))$ by the r-jet extension map (for details & notation, see Abraham 1). But $J^r(E)$ is a G vector bundle over M in a natural way ('J^r' is a functorial construction for VB(M)-Palais 1). We may easily verify that G acts continuously on $C^0(J^r(E))$. Consequently the G-action on $C^r(E)$ is continuous. (Alternatively, just proceed directly).

Thus lemma 1 implies that we may suppose $(C^r(E),\|\ \|_r,G)$ is a G-Banach space. In future, when considering sections of G-vector bundles, we shall always assume that the C^r topology is defined by a G-invariant norm.

If $E \in GVB(M)$, M compact, we have:

<u>Proposition 1</u>

i) $C^r_G(E) \in B_G$

ii) $C^r_G(E)$ splits as a G-Banach subspace of $C^r(E)$.

Proof

$C^r_G(E)$ is clearly a vector subspace of $C^r(E)$, therefore we have only to prove it closed to prove i); but this is obvious since the limit of a convergent sequence of G-invariant sections is G-invariant.

Given $s\varepsilon C^r(E)$, define $Av(s)$ as follows:

$$Av(s)x=\int_G g^{-1}s(gx)dg.$$

We note immediately that $Av(s)\varepsilon C^r_G(E)$.

Define $NC^r_G(E)=\{s\varepsilon C^r(E):Av(s)=0\}$. We note the following:

1. $Av(s)=s$ iff $s\varepsilon C^r_G(E)$-since $Av(s)$ is equivariant.
2. $NC^r_G(E)$ is closed.
3. Any $s\varepsilon C^r(E)$ can be written uniquely as $s_1+s_2, s_1\varepsilon C^r_G(E)$, $s_2\varepsilon NC^r_G(E)$.

Thus we have a split short exact sequence in $B_{\mathbb{C}}$:

$$0\longrightarrow NC^r_G(E)\longrightarrow C^r(E)\xrightarrow{Av} C^r_G(E)\longrightarrow 0,$$

& $C^r(E)=C^r_G(E)\oplus NC^r_G(E)$

Let $E\varepsilon VB(M)$, M compact. Let K be a strictly positive smooth measure on M & let E have a Riemannian structure $\langle\,,\rangle_{E_x}$. Let $L^p(E)$ denote the set of Borel measurable sections s of E s.t.:

$$\|s\|=\left(\int_G\langle s(x),s(x)\rangle^{p/2}dK(x)\right)^{1/p}<\infty.$$

Then we have that $L^p(E)$ with this norm is a Banach space-independent of K & the Riemannian structure on E (see, for example, Palais 1)

If we define $L^2_r(E)=H^r=\{s\varepsilon L^2(E): s \text{ is } C^r\}$ it is well known that H^r is a Hilbert space.

If $E \in GVB(M)$, then one may show, as above, that H^r is a G-Hilbert space. Proposition 1 now follows immediately, since closed subspaces of Hilbert spaces always split.

Remark: The averaging map defined in Proposition 1 is continuous, & therefore C^∞, w.r.t. the C^r norm. The proof of this is direct & easy using the compactness of G.

In fact if $(E, \| \|_G, G)$ is a G-Banach space, then E has an averaging map Av, & Av is continuous in the $\| \|_G$ norm.

Proof

Define $Av(x) = \int_G gx\,dg$, for $x \in E$.

Then:

$$\|Av(x)\|_G = \left\|\int_G gx\,dg\right\|_G \leq \int_G \|gx\|_G\,dg \leq \int_G \|g\|_G \|x\|_G\,dg$$

$$= \left(\int_G \|g\|_G\,dg\right)\|x\|_G.$$

Thus since G is compact & G acts as a group of continuous linear transformations the integral on the second line is bounded proving that Av is continuous.

The aim of the rest of Part 1 is to generalise Proposition 1 to the case when $E \in GFB(M)$, M compact, & to show that $C^r_G(E)$ is a C^∞ submanifold of $C^r_G(E)$.

The method used to obtain this result will be a generalisation of that used in Palais 1 for the G=id case. Although we concern ourselves only with the C^r_G & H^r_G functors, the results are true, with no extra effort, for section functors satisfying the appropriate G-version of Palais' axioms.

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2. Special case: $H^r_G(E)$

Recall from Palais 1, that if $E\epsilon FB(M)$, we define $H^r(E)$ to be the set of all sections s of E, s.t. $s\epsilon H^r(V)$ for some open vector subbundle V of E, i.e.:

$$H^r(E)=\bigcup_V H^r(V).$$

Now, it is known that, if $r>n/2$, $H^r(E)$ may be given a C^∞ manifold structure. Since $H^r(V)$ is Hilbert, it follows that $H^r(E)$ is a C^∞ manifold modelled on Hilbert space for $r>n/2$. $n=\dim M$.

Suppose $E\epsilon GFB(M)$, then we have a natural G-action on $H^r(E)$, $s\longmapsto g.s.g^{-1}$. That this action is differentiable is a consequence of work in Palais 1. Further, if $s\epsilon H^r(E)$, it is easy to see that G(s) is a closed submanifold of $H^r(E)$. Finally we note that $H^r_G(E)$, the equivariant sections in $H^r(E)$, is the fixed point set of the G action.

Now since $H^r(V)$ is Hilbert, it follows from Lang 1, that $H^r(E)$ admits C^∞ p.o.1. & so Riemannian metrics. Consequently we may construct normal bundles of closed submanifolds of $H^r(E)$. If $E\epsilon GFB(M)$, then $H^r(E)$ is also a G-manifold & hence admits an equivariant Riemannian metric, see, for example, Palais' lecture in Borel 1.

As a consequence of the above remarks, we may apply the usual techniques of Lie transformation group theory as given in Palais, Borel 1, to give:

Theorem 1

Let $E\epsilon GFB(M)$. Then $H^r_G(E)$ is a closed C^∞ submanifold of $H^r_G(E)$, when $r>n/2$.

We only give a sketch of the proof of Theorem 1, as the theorem will also follow from the techniques applied to prove the corresponding result for C^r_G: these are presented in detail in the rest of part 1.

3. Some natural constructions on GVB(M) & GFB(M)

(G,M):G-manifold.

1. We have a natural G-action induced on the tangent bundle of M, TM, by:

$G \times TM \longrightarrow TM; (g,v) \longmapsto Dg(v)$, where Dg is the differential of $g:M \longrightarrow M$. With this action $TM \in GVB(M)$, since $\forall g \in G$, we have $p.Dg = g.p$, where $p:TM \longrightarrow M$ is the natural projection.

Proceeding inductively, we have a natural G-action induced on T^rM by that on $T^{r-1}M$, & $T^rM \in GVB(T^{r-1}M)$, $r \geqslant 1$. In particular, $TTM \in GVB(TM)$.

2. If $E \in GVB(M)$, we may form the tensor bundle of type $(r,s):T^r_s(E)$ (For details see Abraham 2). Then if $g:E \longrightarrow E$, we have a map $g^r_s:T^r_s(E) \longrightarrow T^r_s(E)$ s.t.

$$\begin{tikzcd} T^r_s(E) \arrow[r, "g^r_s"] \arrow[d, "\pi^r_s"'] & T^r_s(E) \arrow[d, "\pi^r_s"] \\ M \arrow[r, "g"] & M \end{tikzcd}$$

commutes.

Thus $T^r_s(E) \in GVB(M)$.

3. Suppose now that $E \in GFB(M)$. As, for example, in Palais 1 we may construct the 'vertical tangent bundle' $VT(E)$ over E. $VT(E) \in VB(E)$, & in fact $VT(E)$ is a subbundle of TE, in a natural way. Let $g \in G$, then $g:E_x \longrightarrow E_{gx}$ & so $Dg:T(E_x) \longrightarrow T(E_{gx})$. Thus G induces an action on $VT(E)$ & with this action $VT(E) \in GVB(E)$; in fact $VT(E)$ is a G vector subbundle of TE. The G-action on $VT(E)$ is a restriction of the natural G-action on TE.

Let $q:VT(E) \longrightarrow E$, be the natural projection. We define a subbundle $VVT(E)$ of $T(VT(E))$:

$$VVT(E)=dq^{-1}(VT(E))=\{x\epsilon T(VT(E)):dq(x)\epsilon VT(E)\}.$$

We assert that $VVT(E)\epsilon GVB(VT(E))$.To see this we first make the following remarks:

It is well known that there are two natural vector bundle structures on TTM,regarded as a bundle over TM. First the ordinary tangent bundle structure $p_{TM}:TTM\longrightarrow TM$, & secondly the structure given by the differential of the tangent bundle projection for M. $Dp_M:TTM\longrightarrow TM$ ($p_M:TM\longrightarrow M$).

Now if M is a G-manifold we have already shown that $p_{TM}\epsilon GVB(TM)$,we assert that $Dp_M\epsilon GVB(TM)$,with the natural G-action on TTM.To see this we note:

$$Dp_M.D^2g=D(p_M.Dg)=D(g.p_M)=Dg.Dp_M.$$

Now note that $VVT(E)\subset TT(E)$ & to show that the G-action on TTE restricts to a G-action on VVT(E) it is clearly sufficient to show that if $x\epsilon T(VT(E))$ is s.t. $dq_E(x)\epsilon VT(E)$ then $dq_E(gx)=g.dq_E(x)$.But, by the above remark,this is true. Therefore we have a G-action on VVT(E).Since this action is the restriction of the action on TTE,it is clear that $VVT(E)\epsilon GVB(VT(E))$.

Our aim is now to show that $C^r_G(E)$ may be given the structure of a C^∞ Banach manifold.In the proof given in Palais 1 for the G=id case,one of the basic ideas was that of the construction of a neighborhood of f(M) in E,$f\epsilon C^r(E)$, in such a way that it can be regarded as a vector subbundle of E.Thus one can induce a local additive structure on a nbd. of f(M) in E s.t. C^r maps near f are C^r vectorfields w.r.t. this linear structure.We then note the fact that sections of a vector bundle over a compact space form a Banach space in the C^r topology & thus we are able to construct a chart for f.

To show that $C^r_G(E)$ is a manifold we will construct a vector bundle nbd. of s(M) for each $s \in C^r_G(E)$, s.t. this vector bundle is a G-subbundle of E. As a consequence we may represent equivariant sections of E near f as equivariant sections of a vector bundle. The methods used in the proof parallel those of Palais 1.

4. Equivariant vector bundle neighborhoods

Definition 2

If $E\in GFB(M)$ & $s\in C^0_G(E)$ then an 'equivariant vector bundle neighborhood' (abbreviated GVBN) of s in E is an element $V\in GVB(M)$, s.t. V is an open subbundle of E with the inclusion a G-map & $s\in C^0_G(V)$. Thus $\forall g\in G$ we require

$$\begin{tikzcd} V \arrow[r, "c"] \arrow[d, "g"] & E \arrow[d, "g"] \\ V \arrow[r, "c"] & E \end{tikzcd} \quad \text{to commute.}$$

Our aim in this section is to prove an existance and uniqueness theorem for GVBN.

Let W be a C^∞ manifold & p_W, p_{TW} be the tangent bundle projections for TTW & TW respectively. Recall that a section X of TTW is called a 2nd. order differential equation if:

1. $dp_W(X(v))=v$ $(v\in TW)$.
2. $p_{TW}X(v)=v$ $(v\in TW)$, i.e. X is a section of TTW.

   Let $j\in R$ & $\mathfrak{J}: TW \longrightarrow TW$ denote the map $v \longmapsto jv$.

   We say that X is a spray if in addition:
3. $X(jv)=d\mathfrak{J}(jX(v))$ $(v\in TW)$.

Suppose W is a G-manifold then we have natural G-actions on TW & TTW & if X is a section of TTW we may define $Av(X)$, as in Proposition 1. We will assume from now on that all sprays are C^∞.

Proposition 2

If X is a spray so is $Av(X)$.

Proof

1. We show $Av(X)$ is 2nd. order, i.e. $dp_W(Av(X)v)=v,(v\in TW)$.

$$dp_W Av(X)v = dp_W\int_G g^{-1}X(gv)dg$$
$$=\int_G dp_W g^{-1}X(gv)dg$$
$$=\int_G g^{-1}dp_W X(gv)dg=\int_G g^{-1}.g(v)dg=v.$$

2. Next we prove $Av(X)$ is a spray:

$$Av(X)(jv)=\int_G g^{-1}X(g(jv))dg$$
$$=\int_G g^{-1}X(jg(v))dg,\text{since } G \text{ is linear on fibers.}$$
$$=\int_G g^{-1}d\mathfrak{J}(jX(g(v)))dg=\int_G d\mathfrak{J}g^{-1}(jX(g(v)))dg$$
$$=d\mathfrak{J}_G(j\int_G g^{-1}(Xg(v))dg)$$
$$=d\mathfrak{J}(jAv(X)v).$$

Now if W is paracompact & finite dimensional there exists a C^∞ spray on W (Lang 1). Hence if W is a G-manifold there exists an equivariant spray over W. We remark that since 2nd. order differential equations are never trivial, we always have non-trivial equivariant sections of TTW.

Let $E\in GFB(M)$, we construct $VT(E)\in GVB(E)$ & $TVT(E)\in GVB(VTE)$. We define the concept of 'bundle 2nd. order differential equations & sprays' over E, with values in the subbundle $VVT(E)$ of $TVT(E)$ as in Palais 1:

Definition 3

A C^∞ vector field X on $VT(E)$ is called a bundle 2nd. order differential equation in E if:

1. $dqX(v)=v$ $(v\varepsilon VT(E))$, where $q:VT(E)\longrightarrow E$ is the natural projection.

If, in addition, for all $v\varepsilon VT(E)$ & for all $j\varepsilon R$ we have;

2. $X(jv)=d\mathfrak{J}(jX(v))$, then we call X a bundle spray over E.

If X is such a bundle spray over E, then we may define $Av(X)$ & prove exactly as in Proposition 2 that $Av(X)$ is an equivariant bundle spray over E. We recall from Palais 1 that if E has paracompact fiber & base then a bundle ppray exists over E. Therefore we have:

Proposition 3

If $E\varepsilon GFB(M)$, where fiber(E) & M are paracompact, then E admits an equivariant bundle spray.

Let X be an equivariant spray for E. If $x\varepsilon VT(E)$ let $w_x(t)$ denote the (local) integral curve through x.

Let $D=\{v\varepsilon VT(E):w_v(1)$ is defined$\}$. We note D is open & also that D is G-invariant (Since if w_v is defined on $(a,b)\subset R$ so is w_{gv} $(g\varepsilon G)$, since $gw_v(t)=w_{gv}(t)$).

We define the exponential map $Exp:D\longrightarrow E; v\longrightarrow q(w_v(1))$. Since X is equivariant so is Exp.

We may now state:

Existance theorem for GVBN

Let $E\varepsilon GFB(M)$ be s.t. fiber(E) & M are paracompact. Let $g\varepsilon C^0_G(E)$. Given a neighborhood N of g(M) in E, $\exists$ a GVBN β of g in E with $\beta\subset N$. Moreover if $g\varepsilon C^\infty_G(E)$ we can choose β s.t. g is the 0-section of β.

Proof

We first prove the following G-version of a Lemma in

Palais 1:

Lemma A

Let $E \varepsilon GFB(M)$, where E is paracompact. Let Exp be the exponential map of an equivariant spray over E. We choose an equivariant Riemannian structure for $VT(E)$ & let d_x denote the corresponding G_x-invariant metric in the fiber E_x. Then there are strictly positive equivariant functions α & T on E (Here we take trivial G-action on R) s.t. if $e \varepsilon E_x$ then Exp maps the open disc of radius $\alpha(e)$ in $T(E_x)_e$ C^{∞} G-isomorphically onto a nbd. U of e in E_x which contains all $e' \varepsilon E_x$ with $d_x(e,e') < T(e)$.

Proof

We first note the following lemma:

Lemma B (Palais 1)

Let $E = M \times F$ be a trivial C^{∞} fiber bundle, whose fiber F is a finite dimensional normed vectorspace. Let Exp denote the exponential map of a bundle spray over E. Given $e_o \varepsilon E$, there is a nbd. $O(e_o)$ of e_o in E & an $r > 0$, s.t. for each $e \varepsilon O(e_o)$ Exp maps the ball of radius r about zero in F (F being identified with $VT(E)_e$) C^{∞} isomorphically onto a nbd. of e in F (identified with the fiber containing e) which includes the ball of radius r/2 about e.

We now use Lemma B to verify Lemma A.

Given $e_o \varepsilon E$, it follows from Lemma B that $\exists$ a nbd. $O(e_o)$ of e_o in E & positive numbers $\gamma(e_o)$ & $\delta(e_o)$ s.t. if $e \varepsilon O(e_o) \cap E_x$ then Exp maps the disc of radius $\delta(e_o)$ about the origin in $T(E_x)_e$ C^{∞} isomorphically onto a nbd. of e in E_x which contains all $e'\varepsilon E_x$ with $d_x(e,e')<\gamma(e_o)$.

We assert that this statement remains true if we replace $O(e_o)$ by $G(O(e_o))$, still with the same positive numbers $\gamma(e_o)$ & $\delta(e_o)$. This follows from the commutativity of:

$$\begin{tikzcd} \{y\varepsilon T(E_x)_e : \|y\|<\delta(e_o)\} \arrow[r, "Exp"] \arrow[d, "g"] & \{e'\varepsilon E_x : d_x(e,e')<\gamma(e_o)\} \arrow[d, "g"] \\ \{z\varepsilon T(E_{gx})_{ge} : \|z\|<\delta(e_o)\} \arrow[r, dashed, "Exp"] & \{f'\varepsilon E_{gx} : d_{gx}(ge,f')<\gamma(e_o)\} \end{tikzcd}$$

Note that $\| \;\|$ & $d_.$ & Exp are equivariant.

Now, as in Palais 1, we let $\{V_b\}_{b\varepsilon\Lambda}$ be a locally finite cover of E by relatively compact open sets which refines $\{O(e)\}_{e\varepsilon E}$ & we choose $e(b)$ s.t. $V_b\subset O(e(b))$. We choose a C^∞ p.o.1. $\{H_b\}_{b\varepsilon\Lambda}$ with $\overline{supp}(H_b)\subset V_b$. Then we have an equivariant p.o.1. $\{\bar{H}_b\}_{b\varepsilon\Lambda}$ s.t. $\overline{supp}(\bar{H}_b)\subset G(V_b)$ (See, for example, Wasserman 1-we just set $\bar{H}_b = Av(H_b)$).

Define:

$\alpha = \sum \delta(e(b))\bar{H}_b$

$\tau = \sum F(b)\bar{H}_b$, where $F(b) = \min\{\gamma(e(b')) : V_b \cap V_{b'} \neq \emptyset\}$.

Now given $e\varepsilon E_x$, Exp maps the disc of radius $\delta(e(b))$ about the origin in $T(E_x)_e$ C^{∞} G_x-isomorphically onto a nbd. of e in E_x which contains all $e'\varepsilon E_x$ with $d_x(e,e')<\gamma(e(b))$, provided $e\varepsilon V_b$. Since:

$$\min_b\{\delta(e(b)) : e\varepsilon V_b\} \leq \alpha(e) \leq \max_b\{\delta(e(b)) : e\varepsilon V_b\} \;\&$$

$$\tau(e) \leq \min_b\{\gamma(e(b)) : e\varepsilon V_b\}$$, the lemma follows.

We now introduce some new notation:

Suppose $s\epsilon C_G^{\infty}(E)$, then $s^*VT(E)\epsilon VB(M)$. We denote $s^*VT(E)$ by $T_s(E)$. If we identify M with $s(M)\subset E$, then $T_s(E)=VT(E)|M$. Since $s(M)$ is a G-invariant submanifold of E, it is easy to see that $T_s(E)\epsilon GVB(M)$.

We now return to the proof of the existance theorem:

We choose an equivariant bundle spray & equivariant Riemannian metric for VT(E) & let Exp, α, τ & d_x be as above. By an approximation theorem of Wasserman (Wasserman 1) $\exists k\epsilon C_G^{\infty}(E)$, arbitrarily close to, g s.t.:

1. $\tau(k(x))>\frac{1}{2}\tau(g(x))$
2. $d_x(k(x),g(x))\leq\frac{1}{2}\tau(g(x))$.

If g is C^{∞} we take k=g. Clearly $d_x(k(x),g(x))<\tau(k(x))$. Set $\Lambda=\{v\epsilon T_k(E):\|v\|<\alpha(k(q(v)))\}$ where $q:T_k(E)\longrightarrow M$ is the bundle projection. Note that since we have chosen an equivariant metric Λ is G-invariant.

Let $J:[0,\infty)\longrightarrow[0,1)$ be a C^{∞} diffeomorphism s.t. $J(t)=t$ for t near zero & define an equivariant diffeomorphism $\mathfrak{J}:T_k(E)\longrightarrow\Lambda$ by:

$$\mathfrak{J}(v)=\alpha(k(q(v)))\,\frac{J(\|v\|)v}{\|v\|}\,.$$

Finally, we define $Q=\mathrm{Exp}.\mathfrak{J}$. This is clearly a C^{∞} equivariant fiber preserving isomorphism of $T_k(E)$ onto an open subset β of E, which we make into an open vector subbundle of E by demanding that Q shall be an equivariant vector bundle isomorphism. Conditions 1 & 2 imply, as in Palais 1, that β is a GVBN of g, & by choice of τ we may insist that $\beta\subset N$.

We next consider uniqueness of GVBN's:

Uniqueness Theorem

If $E\varepsilon GFB(M)$, $s\varepsilon C^0_G(E)$ & if V_1 & V_2 are both GVBN's of s, then $V_1 \cong V_2$ by an equivariant vector bundle isomorphism.

Proof

Suppose $H\varepsilon GVB(M)$, then for each $s\varepsilon C^\infty_G(H)$, there is a canonical equivariant isomorphism of $T_s(H)$ with H. This is clear, since we have a canonical identification of $T_s(H)_x$ with H_x-using the identification of $T_s(H)$ with $VT(H)|M$ & the identification $VT(H)_e = T(H_x)_e$ with H_x.

We remark that if E_1 is an open subbundle of E_2, E_1 & $E_2 \varepsilon GFB(M)$, then $VT(E_1) = VT(E_2)|E_1$. Hence, if $s\varepsilon C^\infty_G(E_1)$ then $T_s(E_1) = T_s(E_2)$.

We have the following sub-lemma (Palais 1)

Sub-lemma

If $E\varepsilon GFB(M)$, $s\varepsilon C^\infty_G(E)$ & H is a GVBN of s in E, then H is equivariantly isomorphic to $T_s(E)$.

Proof: Follows from above remarks.

Now suppose V_1 & V_2 are GVBN of $s\varepsilon C^0_G(E)$, then by Wasserman's approximation theorem $\exists k\varepsilon C^\infty_G(V_1) \cap C^\infty_G(V_2)$ & by the above sublemma $V_1 \cong T_k(E)$. q.e.d.

Remark: Palais at this point defines the notion of bundle tubular nbds, it is easy to see that, given the existance of equivariant bundle sprays, one can define & prove existance & uniqueness theorems for bundle tubular nbds.

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5. Differentiability & the differential structure for $C^r_G(E)$

We recall some further notation:

Let E_1 & $E_2 \in GFB(M)$ & $f:E_1 \longrightarrow E_2$ be a GFB-morphism. Then, since f is fiber preserving, it follows that the GVE-morphism $df:T(E_1) \longrightarrow f^*T(E_2)$ maps $VT(E_1)$ into $f^*VT(E_2)$ & we denote by $Vf:VT(E_1) \longrightarrow f^*VT(E_2)$ the corresponding restriction. Clearly Vf is a GVB-morphism & we term Vf the 'vertical differential of f'. Given $s \in C^\infty_G(E_1)$ we define $V_s f = Vf.s$. Then $V_s f:T_s(E_1) \longrightarrow T_{f.s}(E_2)$ is a GVB-morphism called the 'vertical differential of f along s'. We have the following theorem (Palais 1):

Theorem A

If $H,K \in VB(M)$, M compact, & if f is an FB-morphism then $C^r(f):C^r(H) \longrightarrow C^r(K)$ is a C^∞ map & for each positive integer r, $d^r C^r(f) = C^r(V^r f)$, where $C^r(f)s(x) = f(sx)$, s a section of H.

Noting that $C^r_G(W)$ is a closed subspace of $C^r(W)$, $W \in GVB(M)$, & that the inclusion & restriction maps are C^∞ we have:

Theorem 2

If H & $K \in GVB(M)$, M compact, & if f is a GFB-morphism $f:H \longrightarrow K$, then:

$C^r(f):C^r_G(H) \longrightarrow C^r_G(K)$ is a C^∞ map & for each positive integer r, $d^r C^r(f) = C^r(V^r f)$.

We have the following theorem of Palais 1:

Theorem B

E_1 & $E_2 \in FB(M)$ & $f:E_1 \longrightarrow E_2$ is an FB-morphism. Then $f_*:C^o(E_1) \longrightarrow C^o(E_2)$ restricts to a function $C^r(f)$, where $C^r(f):C^r(E_1) \longrightarrow C^r(E_2)$. Moreover, if L_1 & L_2 are open vector subbundles of E_1 & E_2 respectively &

$Q=\{a \in C^r(L_1):C^r(f)(a) \in C^r(L_2)\}$, then Q is open in $C^r(L_1)$ & $C^r(f)$ maps Q C^∞ into $C^r(L_2)$.

Now we note that a GVBN is a VBN & that, if $E \in GFB(M)$, we may construct all VBN's using equivariant sprays & metrics. Further inclusion & restriction maps of Banach subspaces of Banach spaces are C^∞ so we may state as a corollary of Theorem B:

Theorem 2

E_1 & $E_2 \in GFB(M)$ & $f:E_1 \longrightarrow E_2$ is a GFB-morphism. Then $f_*:C^o_G(E_1) \longrightarrow C^o_G(E_2)$ restricts to a function $C^r(f)$, where $C^r(f):C^r_G(E_1) \longrightarrow C^r_G(E_2)$. Moreover if L_1 & L_2 are open G-vector subbundles of E_1 & E_2 respectively &

$Q=\{a \in C^r_G(L_1):C^r(f)(a) \in C^r_G(L_2)\}$ then Q is open in $C^r_G(L_1)$ & $C^r(f)$ maps Q C^∞ into $C^r_G(L_2)$.

As an easy consequence of Theorem 2 we have:

Theorem 3

If M is a compact G-manifold, then for each $E \in GFB(M)$, $\exists$a unique C^∞ differentiable structure for $C^r_G(E)$ s.t. for each open G-vector subbundle H of E, $C^r_G(H)$ is an open submanifold of $C^r_G(E)$. If $f:E_1 \longrightarrow E_2$ is a GFB-morphism of elements of GFB(M), then $C^r(f):C^r_G(E_1) \longrightarrow C^r_G(E_2)$ is a C^∞ map w.r.t.

the above differentiable structure for $C^r_G(E_1)$ & $C^r_G(E_2)$.

Theorem 4

If M is compact, $E \in GFB(M)$, then $C^r_G(E)$ is contained in $C^r(E)$ as a closed submanifold.

Proof

Let $C^r_G(H)$ be a chart for $C^r_G(E)$, then $C^r(H)$ is a chart for $C^r(E)$. Now use Proposition 1.

Theorem 5

Let $E_1 \in GFB(M)$, if $s \in C^{\infty}_G(E_1)$, then $T(C^r_G(E_1))_s$, the tangent space to $C^r_G(E_1)$ at s, can be identified canonically with $C^r_G(T_s(E_1))$. Moreover, if $f: E_1 \longrightarrow E_2$ is a GFB-morphism then the differential of $C^r(f): C^r_G(E_1) \longrightarrow C^r_G(E_2)$ at s, when regarded via the above canonical identification as a linear map of $C^r_G(T_s(E_1))$ into $C^r_G(T_{f.s}(E_2))$ is given by:

$$d(C^r(f))_s = C^r(V_s f)$$

Proof: Essentially as in Palais, with appropriate modifications for G case using previous work.

We conclude Part 1 with a few remarks, the proofs of which follow easily from Palais 1 using the results of Part 1.

1. The inclusion map $i: C^r_G(E) \longrightarrow C^r(E)$ is C^{∞}.
2. Since one can regard C^r-equivariant maps from M into N as equivariant sections of the trivial G-bundle $M \times N \xrightarrow{\Pi} M$, it follows that (M compact) that $\mathfrak{C}^r_G(M,N)$ is a C^{∞} manifold.
3. Let $\mathfrak{Diff}^r_G(M)$ denote the equivariant diffeomorphisms of

M. Then $\mathfrak{Diff}^r_G(M)$ is an open submanifold of $\mathfrak{C}^r_G(M,M)$.

4. Suppose E & E'εGFB(M) & are s.t. G induces the same action on M, then G acts on the fiber product E x_M E' in a natural way & we have:

$$C^r_G(E) \times C^r_G(E') = C^r_G(E \times_M E')$$

5. We may put a differentiable structure on $C^o_G(M,N)$, where M is a compact G-space, not necessarily amanifold. For example, we have equivariant Finsler structures.

Part 2: Equivariant Vector Fields

1. Notations & Preliminaries

The following constitutes a brief summary of some of the elementary definitions & results in the theory of Differentiable Lie group actions on manifolds. A good reference is Palais' lecture in Borel 1.

If (M,G) is a G-manifold & $x\in M$, recall that

$G_x=\{g\in G: gx=x\}$ is called the stabiliser group of G

at $x\in M$ & $G_x\subset G$ as a (closed) Lie subgroup.

If $H\subset G$ as a closed subgroup we define $(H)=\{gHg^{-1}: g\in G\}$ & call sets of the form (H) 'G-orbit types'.

If $v\in M/G$ is an orbit in M, say $v=G(x)$, then, since $G_{gx}=gG_xg^{-1}$, it follows that $\{G_y: y\in v\}=(G_x)$ is a G-orbit type which we will call the orbit type of v & denote by $[v]$.

Two orbits are called <u>equivalent</u> if there exists an equivariant diffeomorphism of one onto the other.

<u>Proposition A</u>

Two orbits are equivalent iff they are of the same type.

<u>Proof</u>: Essentially in <u>Palais, Borel 1</u>.

If (M,G) is a G-manifold, then those subsets of M which are unions of all orbits of a fixed type form a partitioning of M into G-invariant subsets. We call this partitioning of M (or M/G) the orbit structure of M. We have the following result:

<u>Proposition B</u>

If M is a compact G-manifold, then the orbit structure is Finite, i.e.:

$$M = M_1 \cup \ldots \cup M_N,$$

where $M_i \subset M$, consists of all points of type i.

Further each $M_i \subset M$ as a submanifold: not in general closed or connected.

Proof: In Borel 1

Next we define the notion of slice, the definition given here is, for convenience, less general than that in Borel 1.

Definition 1

A slice at x is the 'fiber' at x of a G-tubular nbd. of $G(x)$, i.e. The bundle E, associated to the tubular nbd, $\in GVB(M)$, & the tubular map is equivariant (Lang 1). Generally in the sequel E will be the normal bundle of $G(x)$.

Proposition (Kozul 1: See Borel 1 for details)

A slice at x, S_x, satisfies the following conditions:

1. S_x is G_x-invariant.
2. $gS_x \cap S_x \neq \emptyset$, implies $g \in G_x$.
3. If $q: U \longrightarrow G$ is a local cross-section in G/G_x, then the map $F: U \times S_x \longrightarrow M$, defined by $F(u,s) = q(u)s$ is a diffeomorphism of $U \times S_x$ onto an open set of M.
4. We may choose a coordinate system at x s.t. G_x acts linearly on S_x & S_x is an open disc with center $\{x\}$ in an invariant subspace. G_x in fact acts orthogonally on S_x in a suitable coordinate system.

Theorem 6

If M is a compact G-manifold, then if $M = \bigcup M_i$, each M_i may be written as a finite union $\bigcup_{J_i} M_i^j$, where each M_i^j is a connected submanifold component of M.

For a proof of Theorem 6, see Appendix 1.

If M_i is the set of points of type i & $x \in M_i$, then let $M_{i,x}=\{y \in M_i : G_y = G_x\}$. More generally, if $W \subset M$ as a G-invariant set define $W_x=\{y \in W : G_y = G_x\}$.

Let $M = M_1 \cup \dots \cup M_N$ be the orbit decomposition of M. WE will define a partial order on the set $\{1,\dots,N\}$ as follows:

Say $i<j$ if $\exists x \in M_i, \exists y \in M_j$ s.t. G_x contains G_y as a proper subgroup. This relation is transitive for:

$i<j, j<k$ imply $G_x \supset G_y, G_z \supset G_r$ for some $x \in M_i$, y & $z \in M_j$ & $r \in M_k$. Now $\exists g \in G$ s.t. $g^{-1}G_z g = G_y$. Hence $G_r \subset gG_x g^{-1}$, giving $i<k$.

__Definition 5__

If $x \in M_i$, define type(x)=i.

Every point in M has a unique type

__Proposition 4__

If $i,j \in \{1,\dots,N\}$, then if $i>j$, we may find, for any $x \in M_i$, an open nbd. U_x of x s.t. $G(U_x) \cap M_j = \emptyset$.

If i & j are not related, we may for any $x \in K_i$ find a nbd. U_x of x s.t. $G(U_x) \cap M_j = \emptyset$, & similarly for j.

__Proof__

Let S_x be a slice at x. Then $S_x \cap M_j = \emptyset$-property of slice. Hence $G(S_x) \cap M_j = \emptyset$. Take $U_x = G(S_x)$. Second case is similarly proved.

Suppose $M_I = \bigcup_{i \in I} M_i$, where $I \subset \{1,\dots,N\}$. Then we have:

__Proposition 5__

If M_I Possesses the following property:

" $x \in M_I$ implies all $y \in M$ s.t. type(y)$\leq$type(x) belong to M_I".

Then M_I is closed.

Proof

We show $M - M_I$ open using the argument of Proposition 4.

Remark: In case M is non-compact we still have an orbit decomposition $M = \bigcup_{i \in J} M_i$, where J is not necessarily finite. We may define type( ) & construct a partial order on J as above. With the obvious definitions Proposition 5 still holds.

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2. A Weak G-Transversality Isotopy Theorem

This section is devoted to a proof of a generalisation of Thom's Isotopy theorem to certain situations involving equivariant maps. The definition of 'transversality' given here is less general than that used in the sequel in, for example, the consideration of transversality of stable & unstable manifolds of critical elements of equivariant vector fields.

We consider the set of C^r equivariant embeddings of M into N with the C^r topology (see Part 1), where M is a compact G-manifold & N is a finite dimensional G-manifold. We denote the set of equivariant embeddings by $\mathfrak{E}^r_G(M,N)$.

Since $\mathfrak{E}^r_G(M,N)$ is open in $\mathfrak{C}^r_G(M,N)$, $\mathfrak{E}^r_G(M,N)$ is an open submanifold of $\mathfrak{C}^r_G(M,N)$ & hence a submanifold of $\mathfrak{C}^r(M,N)$. Although the definitions are in terms of embeddings, applications will generally be to $C^r_G(E)$, where $E \in GFB(M)$.

We suppose that W is a G-invariant submanifold of N.

As an example N might be TM, $W=(TM)_0$ & then $\mathfrak{E}^r_G(M,TM) \supset C^r_G(TM)$ as a closed submanifold.

Definition 6

Write $f \pitchfork_{G,x} W$, 'f is weak G-transversal to W at x', $x \in M$, $f \in \mathfrak{E}^r_G(M,N)$ if one of the following occurs:

1. $f(x) \notin W$.
2. $(T_x f)(T_x M) \cap T_{fx} W = T_{fx} G(fx)$.
3. $(T_x f)(T_x M) + T_{fx} W = T_x N$.

We will show that 2. is equivalent to:

2'. $(T_x f)(T_x S_x) \cap T_{fx} W = \{0\}$, where S_x is any slice at x.

First note that $(T_x f)(T_x G(x)) = T_{fx} G(fx)$ (in fact if f is an embedding $G_x = G_{fx}$, otherwise we have $G_x \subset G_{fx}$ & f is then onto $G(fx)$). Thus for any $f \in \mathfrak{C}^r_G(M,N)$, with $fx \in W$, we have:

$$T_x f(T_x M) \cap T_{fx} W \supset T_{fx} G(fx).$$

Since f is an embedding $T_x f : T_x S_x \longrightarrow T_{fx} N$ is injective. We now show 2. is equivalent to 2'.

A. $2 \Rightarrow 2'$

This is so since $T_x f$ is injective & $T_x f(T_x G(x)) = T_{fx}(G(fx))$

B. $2' \Rightarrow 2$.

Obvious, since if $\overline{T_x f} : T_x M \longrightarrow T_{fx} N \xrightarrow{\Pi} T_{fx} W$ (where Π is projection) then: $\dim \operatorname{Ker}(\overline{T_x f}) \geqslant \dim(T_x S_x) = p$, say.

& $\dim(\operatorname{Im}(\overline{T_x f}) \geqslant \dim(G(fx)) = n-p$, & so $\operatorname{Dim}(\operatorname{Im}) = n-p$.

<u>Remark</u>: Let S_x be a slice at x & let $y \in S_x$, then $\exists$ a slice S_y for y s.t. $S_y \subset S_x$.

<u>Proof</u>: Consider G_x restricted to S_x & construct a slice S_y for y which is a G_x-slice, then S_y in M is a G slice for y.

<u>Proposition 6</u>

$\bar{\pitchfork}_G W$ is an open relation for $f \in \mathfrak{C}^r_G(M,N)$, when W is a closed G-submanifold of N, where $\bar{\pitchfork}_G W$ means $\bar{\pitchfork}_{G,x} W$ $\forall x \in M$.

<u>Proof</u>

Consider $L(TM,TN)$-this is the tensor bundle over $M \times N$.

Define $Z \subset L(TM,TN)$ to be the set of all $A \in L(TM,TN)$ s.t. if $x \in M$, $y \in N$ & $A \in L(T_x M, T_y N)$ then one of the following occurs:

1. $y \notin W$.
2. $A(T_x S_x) \cap T_y W = \{0\}$, & A is injective.
3. $A(T_x M) + T_y W = T_y N$.

We assert Z is open.

1. & 1. + 3. are open relations (1. + 3. is ordinary transversality). We note that "A injective" is an open property.

Suppose, therefore, that A satisfies 2., i.e. that $A(T_xS_x)\cap T_yW=\{0\}$. Now we assert that $\exists$a slice $S_x'\subset S_x$, of smaller diameter than S_x, s.t.:

$$A(T_z\bar{S}_x')\cap T_qW=\{0\} \text{ or } \emptyset, \text{ for } z\in\bar{S}_x' \;\&\; q\in N.$$

To see this we note that $T_1W\subset TW$ as a compact submanifold & $T_1\bar{S}_x''\subset TS_x$ as a compact submanifold, where $S_x''\subset S_x$. FuRTher:

$$A(T_z\bar{S}_x'')\cap T_qW=\{0\} \text{ or } \emptyset \text{ iff } T_1W\cap A(T_1\bar{S}_x'')=\emptyset.$$

But $T_1W\cap A((T_1S_x)|\{x\})=\emptyset$, so that, using compactness & continuity, $\exists S_x'\subset S_x$ s.t.:

$$T_1W\cap A(T_1\bar{S}_x')=\emptyset \quad \ldots\ldots\ldots\ldots\ldots \quad A^*$$

Since linear injections are open we may, using essentially the normality property for compact sets & continuity, find a nbd. V of A s.t., if $B\in V$, A^* still holds with B replacing A; therefore Z is open, using above remark.

Let $Q=L(TM,TN)-Z$. Then Q is a closed subset of $L(TM,TN)$.

Now we have a map:

$$k':\mathfrak{C}^r_G(M,N)\longrightarrow\mathfrak{C}^0(M,L(TM,TN)); \text{given by } (k'f)x=T_xf.$$

This map is known to be C^0 (Since it is for G=id case, see Abraham 1, page 47). By our construction $f\pitchfork_{G,x}W$ iff $k'f(x)\notin Q$. We may then use the following:

<u>Lemma (See Abraham 1)</u>

If $W\subset N$ as a closed subset, & M is compact then:

$\mathfrak{C}^r(M,N;W)=\{f\in\mathfrak{C}^r(M,N):f(M)\cap W=\emptyset\}$ is open.

As an immediate corollary we have:

Cor.

$\mathfrak{C}^r_G(M,N;W)$ is open.

Noting that $\mathfrak{C}^r_G(M,N)$ is open we have the result.

We will now state the main theorem of this section:

Theorem 7

Let M & N be G-manifolds with M compact & N finite dimensional. Let W be a closed G-invariant submanifold of N.

For $f\in\mathfrak{C}^r_G(M,N)$, let $W_f=f^{-1}W$.

We suppose $f_o \pitchfork_G W, f_o\in\mathfrak{C}^r_G(M,N)$ then:

1. W_{f_o} is a finite union of connected submanifolds $W^i_{f_o}, i\in P$, & is in fact a submanifold of M.

2. There exists an open nbd. N_{f_o} of f_o in $\mathfrak{C}^r_G(M,N)$ s.t. for $f\in N_{f_o}$, W_f is C^{r-1} isotopic to $\bigcup_{i\in I} W^i_{f_o}$, where I is a subset of P.

i.e. there exists a C^{r-1} diffeomorphism $K_f: M\longrightarrow M$, s.t. $K_f W^i_{f_o}=W^i_f, i\in I$, & K_f is C^{r-1} isotopic to the identity.

Proof

We divide the proof into two parts.

1. W_{f_o} is a finite union of closed connected submanifolds.

Let x be a point of type 2., i.e.

$(T_xf)(T_xS_x)\cap T_{fx}W=\{0\}$. This implies that there exists a slice $\bar{S}_x\subset S_x$, s.t. $f_o(\bar{S}_x)\cap W=f_o(x)$. Consequently $G(x)\subset W_{f_o}$ & G(x) is an isolated subset of W_{f_o} & the union of the finite set of submanifold components of G(x). (G(x) is isolated since $G(\bar{S}_x)$ is a nbd. of G(x) disjoint from $W_{f_o}-G(x)$ in M.)

Let $V=\{x\varepsilon M: x$ is a point of type 2. for $f_o\}$.

We assert that V is closed & V/G is a finite subset of M/G. This is obvious, since V is the union of a set of isolated G-orbits: If V/G is not finite we may construct a sequence $x_i \varepsilon G(y_i) \subset W_{f_o}$, s.t. no two x_i belong to the same G-orbit & $x_i \longrightarrow x$. Now $x \notin V$, since the orbits are isolated, so x is not of type 2., but x is not of type 1. + 3. since 1. + 3. is an open condition. Contradiction, since $f \bar{\pitchfork}_{G,x} W$ by assumption.

We consider M-V. This is open & $f_o|(M-V) \bar{\pitchfork} W$, hence $(f_o|(M-V))^{-1}W$ is a closed submanifold of M-V & hence of M-this follows since V is isolated in W_{f_o}.

Since $(f_o|(M-V))^{-1}W$ is a closed submanifold of M, we may find a tubular nbd. U of $(f_o|(M-V))^{-1}W$ s.t. $\bar{U} \subset M-V$ & hence $\bar{U}$ is disjoint from V in M. 1. is therefore certainly true.

2. Let x be a point of type 2.

Then, as above, we may choose $\bar{S}_x$ s.t. $f_o(\bar{S}_x) \cap W = \{f_o x\}$. We may further choose $S_x^* \subset \bar{S}_x \subset S_x$, L_{f_o} & $\tilde{U}$, where L_{f_o} is a nbd. of f_o in $\mathfrak{C}^r_G(M,N)$, $\tilde{U}$ is a nbd. of fx in W, s.t. if $f \varepsilon L_{f_o}$ then either $f(S_x^*) \cap W = \emptyset$, or $f(S_x^*) \cap W = \{q\}$, where $q \varepsilon \tilde{U}$.

This we may do since we may assume f is sufficiently C^r close to f_o & we have the assumption of $\bar{\pitchfork}_G$.

We consider the case when $f(S_x^*) \cap W = q$.

We assert that $G_q = G_x$, for let $Q = f^{-1}q$ .Since f is an embedding Q consists of one point. Now if $G_q \neq G_x$ then G_q is a subgroup of G_x & therefore $G_q(Q)$ is an orbit in S^*_x consisting of more than one point. But if this is so then, since f is equivariant, $f(G_q(Q)) \subset f(S^*_x) \cap W = \{q\}$. Contradiction, since f is 1:1, so therefore $G_q = G_x$.

As a consequence of the above, under perturbation in L_{f_o}, a G(x) orbit in $W_{f_o} \cap V$ either vanishes of perturbs to an equivalent G-orbit, C^0 close to G(x).

We construct a family of C^∞ tubular nbd. pairs (H_i, J_i) indexed by V/G possessing the following properties:

1. H_i & J_i are tubular nbds of G(x), a G-orbit of V, with $H_i \supset \bar{J}_i \supset J_i$.
2. The tubular nbd. pairs are mutually disjoint.
3. (H_i, J_i) are disjoint from some C^r-tubular nbd. K of $(f_o|(M-V))^{-1}W$ in M.

We may further choose our nbd. pairs so that we may find a nbd. N' of f_o in $\mathfrak{C}^r_G(M,N)$ s.t. if $f \in N'$ then either $H_i \cap W_f = \emptyset$ or $H_i \cap W_f = G(y)$, where $G(y) \subset J_i$ & $G_y = G_x, y \in H_x$.

We now restrict our attention to K, which is disjoint from (H_i, J_i).

We may find a tubular nbd. pair of $(f_o|(M-V))^{-1}W$, say (A,B), s.t. $K \supset \bar{A} \supset B$. We may then find a nbd. N" of f_o s.t. if $f \in N"$ then $(f|(M-V))^{-1}W \subset B$ & show, as for example in Abraham'S presentation of Thom's Isotopy theorem in Abraham 1, that there is a C^{r-1} isotopy between $W_{f|(M-V)}$ & $W_{f_o|(M-V)}$ which is supported on B.

We may easily construct a C^∞ isotopy Z_i, supported on J_i for each $f\varepsilon N'\cap N''\cap L_{f_o}$, s.t. if $H_i\cap W_f=\emptyset$ Z_i=id, &, if $H_i\cap W_f=G(y)$, Z_i is an isotopy between $G(x)$ & $G(y)$. To construct Z_i in the second case it is enough to construct an isotopy between x & $G(y)\cap H_x$ (Supposing that (H_i,J_i) is a G-normal bundle pair).

We may choose $N_{f_o}\subset N'\cap N''\cap L_{f_o}$ s.t. if $f\varepsilon N_{f_o}$ then $W_f\cap(M-\bigcup H_i-A)=\emptyset$.

The theorem then follows by piecing together all the various isotopies, which have disjoint supports.

Remark: It is not hard to show in the above that we may require the isotopy constructed to be equivariant.

Cor 7.1

If $f\pitchfork_G W$ & all points $x\varepsilon M$ are of type 1. or 2. we may make our isotopy C^∞

Suppose (M,G) is a G-manifold.

Definition 7

If $s\varepsilon C^r_G(TM)$, we call s an 'equivariant vectorfield'

Let us take, with the preceeding notation, $N=TM$, $W=(TM)_o$.

Definition 8

If $s\varepsilon C^r_G(TM)$ & $s\pitchfork_G TM_o$ we say s is a O-generic vector-field.

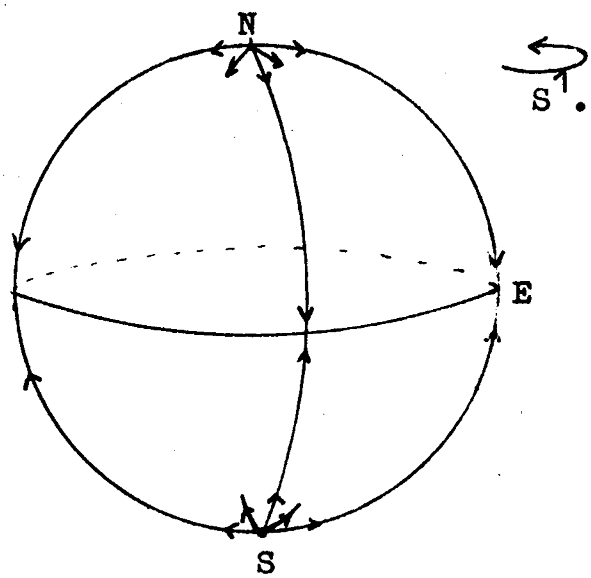
Example 1

$M=S^2$

$G=S^1$, with the action being defined as rotation about the N-S axis-see Diagram 1.

For s we take the equivariant vectorfield described by Diagram 1:N & S are sources & the equator,E, is a sink.It is not difficult to see that we may take s to be 0-generic.

Diagram 1



We see that under perturbation we may change the fixed set E into a closed orbit by introducing a perturbation of s along the direction of E,supported on a nbd. of E.

In the preceeding terminology $W_s=\{N,S,E\}$& under perturbation to s' we can get $W_{s'}=\{N',S'\}$,we note that N & S do not vanish under perturbation.We will return to this point later.

3. Some results about the structure of $C^r_G(E)$ & $\mathfrak{C}^r_G(M,N)$

Definition 9

If $x \in M, y \in N$ we write:

xGy if $\exists h \in \mathfrak{C}^r_G(M,N)$ s.t. $hx=y$.

If $A \subset N$ we write:

xGA if xGy $\forall y \in A$.

Proposition 7

A necessary condition for xGy is $G_y \supset G_x$. If $\exists h \in \mathfrak{C}^r_G(M,N)$ s.t. $hx=y$, then $G_y = G_x$.

Proof: Trivial.

Lemma 2

Let $E \xrightarrow{p} M$, be a G-vector bundle.

Let $x \in M$ & suppose $Z_x \in E_x$ is s.t. $G_{Z_x} = G_x$. Then there exists a section Q of E s.t.:

1. Q is equivariant.
2. $Q(x) = Z_x$.

Further, if type(x)=i, we may choose Q s.t.:

3. $Q(y)=0$, if type$(y) \not\geq i$.

Proof

Let $K = N(G_x)S_1$, where $S_x = S_1 \times S_2$ is a slice at x, see Appendix 1 for notation.

Let $E_K = i^*E$, be the restriction of E to the manifold K:

$$\begin{array}{ccc} E_K = i^*E & \longrightarrow & E \\ \downarrow p & & \downarrow p \\ K & \xrightarrow{i} & M \end{array}$$

i: inclusion of K in M.

Now E_K is a G_x vector bundle over K, s.t. $G_x = id$ on K.

We also note that $S_1 \subset K$ as a submanifold.

Let S_x^* be a slice at x (in M) s.t. $\bar{S}_x^* \subset S_x$, we assert that, to prove the Lemma, it will be sufficient to find $Q' \varepsilon C^r_{G_x}(E_K|S_1)$ s.t. $Q'(x)=Z_x$, $Q'=0$, outside $S_1^* \subset S_1$:

Suppose we have such a Q', we extend Q' to $Q^* \varepsilon C^r_G(E|G(S_1))$ by setting:

$$Q^*(gy)=gQ'(y), y \varepsilon S_1, g \varepsilon G.$$

This is well defined & easily shown to be C^r, if Q' is C^r. It is easily shown to be equivariant.

Now, using Whitney's extension theorem, we may extend $Q^*|G(S_1^*)$ to Q'' on M, making $Q''=0$, outside of $G(S_x^!)$, where $S_x^!$ is chosen s.t. $S_x \supset \bar{S}_x^! \supset S_x^*$ & we also have Q'' to be C^r.

Finally define $Q=Av(Q'')$. Then it is easily checked that Q is C^r, equivariant & $Q|S_1^*=Q'$. Hence $Q(x)=Z_x$. Further, by construction, $Q(y)=0$ if type$(y) \not> i$. This is so since $G(S_x^!)$ is a nbd. of the type refered to in Proposition 4. Thus it suffices to construct Q'.

Let $Av:E_K \longrightarrow E_K$ be the map defined by:

$$Av(e)=\int_{G_x} g(e)dg.$$

Av is clearly a vector bundle morphism (for example, Atiyah 1, page 37). If we denote Ker(Av) by L, we see that L is a G_x-subbundle of E_K. Letting S denote the orthogonal complement of L w.r.t. a G_x-invariant Riemannian metric on E_K, we see that S may be given the structure of a G_x subbundle of E_K. But $S \cong Im(Av)=\{z \varepsilon E_K : G_x=G_z\}$.

We restrict our attention to the bundle $S \xrightarrow{p} K$, G_x=id on S. Now $Z_x \varepsilon S$, choose a C^∞ function $A:S_x'' \cap K \longrightarrow R$, where

1. $\bar{S}''_x \subset S_x$
2. $A=0$, outside of $\bar{S}^*_1 \subset S''_1$, & $A=1$ on a nbd. of x in S_1.

We now define $\bar{Q}'$ on S_1 by setting $\bar{Q}'(y)=A(y)Z_x$-in some suitable trivialisation about x for S-we put $\bar{Q}'=0$ elsewhere on S_1. $\bar{Q}'$ then clearly defines a C^∞ section of $S|S_1$. But $\bar{Q}'$ clearly extends, by inclusion, to an element of $C^r_{G_x}(E_K|S_1)$.

As an aside we conclude this section with a proposition giving some insight into the structure of $J^0_G(M,N) \subset J^0(M,N)$. We will not use this result elsewhere in this paper, though it may perhaps give some motivation for the definitions in the next section.

Proposition 8

M,N: G-manifolds with M compact & N finite dimensional.

Let $x \in M, y \in N$ & xGy.

Define $N^*_x = \{z \in N : G_z \supset G_x\}$.

Let N_x be the connected component of N^*_x containing y, then xGN_x.

Proof

Let $E \xrightarrow{p} M$ be a G-fiber bundle.

We know that $C^r_G(E)$ is a C^∞ differentiable manifold. We first make the following remark about the coordinate structure defined on $C^r_G(E)$ in Part 1.

Let K be any (relatively) compact G-invariant subset of E. Then, given an equivariant Riemannian structure on $VT(E)$, there exists a $p>0$, s.t. whenever $f \in C^r_G(E)$ & $f(M) \subset K$, there exists a GVBN of $f(M)$, N_f say, defining a coordinate chart, s.t. $N_f \cap E_x$ contains an open disc in E_x of diameter at least $2p$, & D contains the disc in E_x of radius p about fx. Further p may be chosen to be independent of f & x, provided $f(M) \subset K$.

That this may be done is immediate from the proof of the existance of GVBN's of f(M), i.e. in Lemma A α & γ have strictly positive infimums on compact sets.

We may regard $\mathbb{C}^r_G(M,N)$ as $C^r_G(M, M \times N)$, where $M \times N$ is the trivial bundle over M. If N is not compact we construct a family of compact subsets N_i of $M \times N$ s.t.:

1. $N_0 = Im(\bar{h})$, where $\bar{h} = (id, h): M \longrightarrow M \times N$ & h is the map occuring in Definition 9.
2. $\bigcup_i N_i = M \times N$.
3. $N_0 \subset \ldots \subset N_i \subset N_{i+1} \subset \ldots$
4. N_i is G-invariant.

We note that we may certainly construct such a family satisfying 1,2 & 3, to get 4. we just have to construct N_i using an equivariant metric, which exists since N is paracompact.

For each N_i we have a corresponding $p_i > 0$, as above.

If N is compact we just take $N_i = M \times N$.

If M & N are connected we could assume above that the N_i are connected. In any case if A & B are the connected components of x & y in M & N respectively we can insist that:

5. $N_i \cap (A \times B)$ is connected.

With the notation of 1. above, we take a coordinate nbd. N_h of $\bar{h}$ in $C^r_G(M, M \times N)$. We choose N_h s.t. $N_h = C^r_G(E)$, where

E is a GVBN of $\bar{h}(M)$ in $M \times N$. Since $\bar{h}$ is injective $G_x = G_{(x,y)}$.

Lemma 2 implies that if we are given $Z_x \varepsilon E_x$, with $G_{Z_x} = G_x$, then $\exists Q \varepsilon C_G^r(E)$ with $Q(x) = Z_x$. But a point $Z_x \varepsilon E_x$ corresponds to a point $(x,y') \varepsilon M \times N$ s.t. $G_{y'} \supset G_x$. Thus $\forall (x,y') \varepsilon E_x$ with $G_{y'} \supset G_x$ $\exists f \varepsilon C_G^r(M, M \times N)$ s.t. $fx = y'$.

Consequently, if xGy then xGW, where W is an open subset of N_x.

Set $W_i = W \cap N_i$. We shall prove that W_i is closed in N_i, which since N_i is connected (i.e. $N_i \cap (A \times B)$ is connected) implies that $N_i = W_i$ & hence $W = N_x$.

Let $z \varepsilon \partial_{N_i} W_i$. Then $\exists$a sequence of points of W_i, y_n, s.t. $y_n \longrightarrow z$.

For each y_n, $\exists \bar{h}_n \varepsilon C_G^r(M, M \times N)$ s.t. $h_n x = y_n$. Now $\exists N_0$ s.t. for $n \geqslant N_0$ $d(y_n, z) < p_i$. Choose $\bar{h}_m$, where $m \geqslant N_0$. Then if E is a GVBN of $\bar{h}_n(M)$, of the type refered to at the begining of the proof, we have $z \varepsilon E$ & $G_z = G_x$. Comsequently, by Lemma 2, $\exists h \varepsilon \mathbb{C}_G^r(M,N)$ s.t. $hx = z$. Therefore $z \varepsilon W_i$ & $W_i = N_i$.

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4. A Transversality Lemma

We know that for maps satisfying the conditions of Definition 6, that $\overline{\pitchfork}_G W$ is an open condition, if W is a closed G-invariant submanifold of the image space. From now on we shall consider $C^r_G(E)$, rather than $\mathfrak{C}^r_G(M,N)$, where E is a G-fiber bundle over compact M & W is a closed G-invariant submanifold of E.

This section is devoted to proving a technical lemma,

which will eventually enable us to show that, for certain W, the subset of $C^r_G(E)$ satisfying a condition, somewhat stronger than $\overline{\pitchfork}_G W$, is not only open but also dense.

Let T be a subset of M, we have the evaluation map:

$$ev: C^r_G(E) \times T \longrightarrow E, (f,t) \longmapsto f(t).$$

If T is a C^r submanifold of M, we know (see, for example, Abraham 1) that ev is C^r.

Let W be a compact G-invariant submanifold of E. Let $W = W_1 \cup \ldots \cup W_m$ be the orbit decomposition of W, which is finite.

We have a locally finite orbit decomposition of E, $E = \bigcup_{i \in I} E_i'$, we see that for each $j \in \{1,..,m\}$ $W \supset W_j = E_i' \cap W$ for some $i \in I$. As previously described we have a partial order $\succ$ on the orbit types of M & E. Let $M = M_1 \cup \ldots \cup M_N$ be the orbit decomposition of M. Suppose the orbit type of $M_i = (G_x)$ for some $x \in M_i$.

Define $\mathfrak{h}_i = \{z \in M_i : G_z = G_x\}$, $\mathfrak{h}_i$ is a submanifold of M_i since it is the fixed point set of the induced G_x action on M_i. Consider:

$$ev: C^r_G(E) \times \mathfrak{h}_i \longrightarrow E$$

We note that ev takes values in $\mathfrak{S}_i$, where $\mathfrak{S}_i=\{y\varepsilon E:G_y=G_x\}$. $\mathfrak{S}_i$ is a submanifold of E, thus: $ev:C^r_G(E)\times\mathfrak{h}_i\longrightarrow\mathfrak{S}_i$ is a well-defined C^r map.

Now $\mathfrak{h}_i$ is a finite union of submanifolds of M (see Appendix 1) thus $\mathfrak{h}_i=\bigcup_{j\varepsilon J_i}\mathfrak{h}_i^j$. We will suppose $x\varepsilon\mathfrak{h}_i^j$ & restrict attention to $\mathfrak{h}_i^j$.

Now take $(f,x)\varepsilon C^r_G(E)\times\mathfrak{h}_i^j$. Let $\mathfrak{f}$ be a GVBN of $f(M)$ in E. Then $C^r_G(\mathfrak{f})$ is a C^∞ chart for f in $C^r_G(E)$. We observe that, as in Lemma 2, $\mathfrak{S}_i\cap\mathfrak{f}=\mathfrak{W}$, may be regarded as a vector subbundle of $\mathfrak{f}$.

Let U be a trivialising coordinate nbd. of x for $\mathfrak{W}$. We will study $T_{(f,x)}ev$. Locally we have, letting $U\times\mathfrak{W}_x$ be a trivialisation of $\mathfrak{W}$ at fx, that:

$$T_{(f,x)}ev:C^r_G(\mathfrak{f})\times T_xU\longrightarrow T_xU\times T_{fx}\mathfrak{W}_x;(s,h)\longmapsto(h,s(x)+Df(x)h).$$

Since $T_{fx}\mathfrak{W}_x\approx\mathfrak{W}_x$, we see immediately from Lemma 2, that $T_{(f,x)}ev$ is onto $\forall f\varepsilon C^r_G(E),\forall x\varepsilon\mathfrak{h}_i^j$.

Further $T_{(f,x)}ev$ splits (We prove this as done, for example, in Abraham 1).

In the terminology of Abraham 1, "ev is transversal to points".

Now $W\cap\mathfrak{S}_i=\{z\varepsilon W:G_z=G_x\}$ is a submanifold of W. We set $W_{\mathfrak{S}_i}=W\cap\mathfrak{S}_i$.

Thus we have shown that:

$$ev_i^j\pitchfork W_{\mathfrak{S}_i},\text{ where } ev_i^j=ev\,|\,(C^r_G(E)\times\mathfrak{h}_i^j).$$

Now $W_{\mathfrak{S}_i}$ is a finite union of connected submanifold components, say $W_{\mathfrak{S}_i}=\bigcup_{k\varepsilon K_i}W_i^k$.

We define $E_i^{j,k}$ as follows:

$E_i^{j,k}=\{f\in C_G^r(E):(f|\mathfrak{h}_i^j)\pitchfork W_i^k \text{ in } \mathfrak{S}_i\}$.

Now let $\operatorname{codim}_{\mathfrak{S}_i} W_i^k=p_i^k$, $\dim\mathfrak{h}_i^j=q_i^j$. Where by $\operatorname{codim}_{\mathfrak{S}_i} W_i^k$ we mean the codimension of W_i^k in that component of $\mathfrak{S}_i$ containing W_i^k. Then, provided $r>\max(0,q_i^j-p_i^k)$, we may apply Thom's Transversality density theorem, as stated in Abraham 1, to give:

Lemma 3

$E_i^{j,k}$ is a dense subset of $C_G^r(E)$.

We add some additional remarks about this result, which give some insight into the type of perturbation occuring above.

Let $x\in M_i$ & let S_1 be a slice at x (in M_i). There are two possibilities:

1. $\dim S_1=\dim(N(G_x)S_1)$
2. $\dim S_1<\dim(N(G_x)S_1)$.

As above we may consider $ev:C_G^r(E)\times S_1\longrightarrow\mathfrak{S}_i$, where $\mathfrak{S}_i$ is defined as before. We define a set $E_i^!$ by:

$E_i^!=\{f\in C_G^r(E):(f|S_1)\pitchfork(W\cap\mathfrak{S}_i) \text{ in } \mathfrak{S}_i\}$ & $E_i^!$ is dense in $C_G^r(E)$ under suitable conditions as before.

Let us consider as an example $E=TM\longrightarrow M$ & $W=(TM)_0$. Then if i is minimal (see section 7 for definition) $(TM)_i=TM_i$. Let us assume for simplicity that M_i is connected, then the condition for $E_i^!$ becomes:

$$(f|S_1)\pitchfork(TS_1)_0 \text{ in } T(N(G_x)S_1).$$

Put $N(G_x)S_1=Z$. Now if $\dim(S_1)=\dim(Z)$, we have $2\dim(S_1)=\dim(Z)$ & so if $f\in E_i^!$, $f(S_1)$ may meet $(TS_1)_0$ & $(f|S_1)^{-1}(TS_1)_0$ is a set of codim 0.

On the other hand, if $\dim Z > \dim S_1$, then the $\bar{\mathfrak{m}}$ condition implies that $f(S_1) \cap (T(S_1)_0) = \emptyset$. That this is reasonable is seen by the following remark:

If $\dim Z > \dim S_1$, we may define a non-zero vector field X, supported on a nbd. of $G(x)$, with $X|G(x)$ tangent to $G(x)$. Consequently, if we have a point of intersection of $f(S_1) \cap (T(S_1)_0)$, we may add a perturbation of the above type & turn the fixed point set into a closed orbit of high period, or perhaps a dense orbit on some torus; these points will be discussed in detail later.

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5. Some Perturbation Theory

In this section we restrict attention to $\mathfrak{Diff}^r_G(M)$ & $C^r_G(TM)$, M compact.

Suppose $f \in \mathfrak{Diff}^r_G(M)$ & $fx=x$, then we know that $f(gx)=gx$, using the equivariance of f. Thus $G(x)$ is a fixed set for f.

Let us take a G-normal bundle of $G(x)$ & let $U \times E$ be a trivialisation of this bundle at x.

W.r.t. this trivialisation, the 1-jet of f at x, $j^1f(x)$ has the form:

$$\begin{pmatrix} I_r & X \\ 0 & Y \end{pmatrix}, \text{where } r=\dim U, Y:E \longrightarrow E, X:E \longrightarrow T_xU.$$

Definition 10

We say f is '1-generic at x' iff, with the above notation Y is generic, i.e. Y has no eigenvalues modulus 1.

Equivalently: f is 1-generic at x iff $j^1f(x)$ has r eigenvalues equal to 1 & $\overline{n-r}$ eigenvalues modulus not equal to 1. Thus the definition is obviously independent of the particular representation of $j^1f(x)$.

Noting that $G(x)$ is f-invariant & hence $T_xG(x)$ is T_xf invariant ($f=id$ on $G(x)$) we have a map N_xf induced on the quotient $T_xM/T_xG(x)$. $N_xf:T_xM/T_xG(x) \longrightarrow T_xM/T_xG(x)$.

Then f is 1-generic iff N_xf has no eigenvalues modulus 1.

We make the observation that "f is 1-generic at x" is a stronger condition than '$f \pitchfork_{G,x} \Delta(M)$'.

Definition 11

We say f is 1-generic iff all fixed points of f are 1-generic.

We remark that if G=id, then 1-generic is equivalent to ordinary fixed point genericity, G1, as described, for example in Smale 1, or Abraham 1.

Suppose $X\varepsilon C^r_G(TM)$ & $X(x)=0_x$. Then, as above, $X(gx)=0_{gx}$. Thus G(x) is a singular set for X. Taking a trivialisation at x, of a G-normal bundle of G(x), say $U \times E$, we may suppose that $U \times E=R^r \times R^{n-r}$. W.r.t. this chart,

$X:R^r \times R^{n-r} \longrightarrow TR^r \times TR^{n-r}=(R^r \times R^{n-r}) \times (R^r \times R^{n-r})$ &

$X(u,s)=((u,s);S(u,s))$.

Now $S(u,s)=DS(0,0)+G(u,s), G(u,s)=o(u,s)$ &

$$DS(o,o)=\begin{pmatrix} L & X \\ M & Y \end{pmatrix}, L\varepsilon L(R^r,R^r), Y\varepsilon L(R^{n-r},R^{n-r}) \text{ etc.}$$

But if $S(0,0)=0$, we certainly have $S(u,0)=0$, since X is equivariant. i.e. $L=0, M=0$. Thus:

$$DS(0,0)=\begin{pmatrix} 0 & X \\ 0 & Y \end{pmatrix}$$

Definition 12

With the above notation we say X is 1-generic at x iff Y is generic, i.e. Y has no eigenvalues real part zero.

Equivalently: Iff $j^1X(x)$ has precisely r eigenvalues equal to zero & $\overline{n-r}$ eigenvalues real part non-zero.

Equivalently: Let F_t denote the flow of X, then G(x) is is left fixed by F_t. As in the diffeomorphism case we define $N_xF_t: T_xM/T_xG(x) \longrightarrow T_xM/T_xG(x)$. Then x is 1-generic iff for some $t \neq 0$ N_xF_t has no eigenvalues modulus 1.

We again remark that 'X 1-generic at x' is a stronger condition than '$X \bar{\pitchfork}_{G,x} (TM)_o$'.

Definition 13

We say X is 1-generic iff it is 1-generic at every singular point.

Again we make the observation that this is a generalisation of the G1 property for vectorfields for G=id.

Before starting on the perturbation theory of this section we introduce some definitions from Pugh-Hirsch-Shub 1 (Abbreviated in future to P-H-S 1) which we use later. These definitions are adapted here to the 'G-category'.

Definition 14 (P-H-S)

Let V be a C^1 compact submanifold of a G-manifold M. Let $f \in \mathfrak{Diff}^r_G(M)$. We shall suppose that V is both G-invariant & f-invariant, M is Riemannian-not necessarily G-Riemannian.

We say that f is G-normally hyperbolic at V iff the tangent bundle of M, restricted to V, splits into three continuous subbundles:

$$T_VM = TV \oplus N^u \oplus N^s,$$

invariant by the differential Tf, s.t.:

1. $\sup_{x \in V} \| Tf | N^s(x) \| < \inf_{x \in V} (m(Tf | T_xV))$, & Tf contracts N^s.

2. $\inf_{x\in V}(m(Tf|N^u(x)))>\sup_{x\in V}\|Tf|T_xV\|$, & Tf expands N^u.

Recall that $m(A)=\inf\{|Ax|:|x|=1\}=\|A^{-1}\|^{-1}$.

We say that f is G,r-normally hyperbolic at V if,in addition:

1_r. $\sup\|Tf|N^s\|<\inf(m(Tf^k|TV))$ $1\leq k\leq r$.

2_r. $\inf(m(Tf|N^u))>\sup\|Tf^k|TV\|$ $1\leq k\leq r$.

It is clear that G-normal hyperbolicity implies normal hyperbolicity.

Clearly TV is a G-vector bundle over V & in fact we have:

Proposition 9 (See also Lemma 10,page 126)

With the above notation $TV\oplus N^s\oplus N^u$ is a G-invariant splitting,i.e. N^u & N^s are both G-vector bundles over V.

Proof

Consider condition 1. above:

$$\sup_{x\in V}\|Tf|N^s(x)\|<\inf_{x\in V}(m(Tf|T_xV)).$$

This implies that $\exists$a & b,strictly positive,s.t.:

$$a<b<1$$

$$\|(T^nf|N^s(x))e^s\|<a^n\|e^s\| \qquad\ldots\ldots\ldots\ldots 1^a$$

$$\|(T^nf|T_xV)e^v\|>b^n\|e^v\| \qquad\ldots\ldots\ldots\ldots 1^b$$

Where 1^a & 1^b hold for all $x\in V$ & a & b are independent of x.

Now let us take an equivariant norm on T_VM,say $\|\ \|_G$, by averaging the original metric on M.Since V is compact, $\|\ \|_G$ is equivalent to $\|\ \|$ on V,i.e. $\exists$l,m s.t.:

$$l\|\ \|_G<\|\ \|<m\|\ \|_G.$$

Thus conditions 1^a & 1^b imply:

$$\|(T^n f|N^s(x)e^s)\|_G < (m/l)a^n\|e^s\|_G \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 1_G^a$$

$$\|(T^n f|T_x V)e^v\|_G > (l/m)b^n\|e^v\|_G \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 1_G^b$$

We have similar conditions corresponding to condition 2. We set $m/l=p$, $l/m=q$, for convenience.

Let $(o,v,o)\in(TV\oplus N^s\oplus N^u)_x$. Suppose $g(o,v,o)=(v_1,v_2,v_3)$, belongs to the fiber over gx. Then, using the equivariance of $\|\ \|_G$ & f, we have for $p>0$, & any c, s.t. $a<c<b$:

$$\frac{\|T^p fg(o,v,o)\|_G}{c^p} = \frac{\|T^p f(v_1,v_2,v_3)\|_G}{c^p} \longrightarrow \infty \text{ as } p\longrightarrow\infty \text{ if}$$

$v_3\neq 0$, since $c<1$ & $v_3\in N^u$, or if $v_1\neq 0$, since $c<b$, & use 1_G^b.

$$\Vert$$

$$\frac{\|T^p f(o,v,o)\|_G}{c^p} \longrightarrow 0, \text{ as}$$

$p\longrightarrow\infty$, using 1_G^a & $a<c$.

Thus N^s is a G-vector bundle over V, similarly we may show N^u is a G-vector bundle over V.

With the above definition of G,r-normal hyperbolicity, we have the following:

Proposition 10

If $G(x)$ is a 1-generic fixed set for $f\in\mathfrak{Diff}_G^r(M)$, then f is G,r-normally hyperbolic at $G(x)$.

Proof

Easy: Just show that the tangent space at some point $x\in G(x)$ splits as: $T_xM=T_xG(x)\oplus N_x^u\oplus N_x^s$ & then use the equivariance & the G-action to define the splitting of $T_{G(x)}M$.

Next we consider normal hyperbolicity for flows.

Definition 15 (P-H-S 1, adapted)

Let $\{f^t\}$ be the equivariant flow of $X\in C_G^r(TM)$. We suppose

V is a G-invariant compact C^1-submanifold of M left invariant by the flow.

We say the flow (or X) is G,r-normally hyperbolic at V, if for some individual map f^t ($t \neq 0$) f^t is G,r-normally hyperbolic at V.

We have the following theorem of P-H-S 1:

Theorem

If one f^t is G,r-normally hyperbolic at V, then they all are, except f^0, the identity. The splitting is independent of t.

Proposition 11

If G(x) is a 1-generic singular set for $X \epsilon C^r_G(TM)$, then X is G,r-normally hyperbolic at G(x).

Proof: Easy as in Proposition 10.

We also remark (P-H-S 1) that the splittings in definitions 14 & 15 can be shown to be unique.

We now return to consider some local perturbation theory for equivariant diffeomorphisms & vectorfields.

Lemma 4

Let $f \epsilon \mathfrak{Diff}^r_G(M)$, & suppose G(x) is a fixed set for f. Then there exists an $f' \epsilon \mathfrak{Diff}^r_G(M)$, arbitrarily C^r close to f, s.t. f' is 1-generic on G(x).

Proof

First we note that if M is a G-Riemannian manifold then M is a G_x-Riemannian manifold, since (M, G_x) is a G_x

manifold, by restriction of the G-action.

Now x is a fixed point of the G_x-action on M. T_xM is a G_x-normal bundle of x, w.r.t. a G-invariant metric. If C is a suitable equivariant compression (see, for example, page 17) then $Exp.C: T_xM \longrightarrow N$, where N is an open G_x-invariant disc nbd. of x in M & $Exp.C$ is a G_x-invariant diffeomorphism. Now using $Exp.C$ we may induce a linear structure on N, s.t. G_x acts on N as a group of orthogonal linear transformations-this is Bochner's theorem.

Let $W_xG(x)$ be the orthogonal complement of $T_xG(x)$ in T_xM. We note here that $T_xG(x)$ is G_x-invariant, since $G(x)$ is G_x-invariant & x is a fixed point for x, so, in particular, $W_xG(x)$ is G_x-invariant.

Then $Exp.C(W_xG(x))=S_x$ is a G-slice at x, which may be regarded as a G_x-invariant vector subspace of N. That S_x is a slice is clear since Exp is G-invariant & C may be extended to $T_{G(x)}M$ in a G-invariant fashion, using G, thus S_x defines a G-tubular nbd. of $G(x)$.

Thus identifying N & T_xM using $Exp.C$, setting $T_xM=R^n$, we have $S_x=\{0\} \times R^{n-p}$ & $T_xG(x)=R^p \times \{0\}$, where $p=\dim(G(x))$.

Now there exists an open disc nbd. D of the origin in R^n, s.t. the local representitive f^* of f maps D into R^n. This is so since $\{0\}$ is a fixed poit for f^*.

So for $z\in D$, we have using Taylor's expansion:

$$f^*(z)=Mz+G(z), \text{where } G(z)=o(z) \ \& \ M=Df^*(0)\in L(R^n,R^n)$$

Now suppose $A\in G_x$, we have $A(M\cdot+G(\cdot))A^{-1}z=Mz+G(z)$.

i.e. $AMA^{-1}z+AG(A^{-1}z)=Mz+G(z)$

Thus we have (see, for example, Dieudonne 1, page 143):

$$AMA^{-1}=M, \forall A \in G_x . \quad \ldots \quad 1.$$

Consequently G is equivariant, i.e. $AG(A^{-1}z)=G(z)$.

Now $R^n=R^p \times R^{n-p}$ & G_x respects this splitting, consequently a typical element $g \in G_x$ may be represented as:

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}, \text{where } A \in L(R^p,R^p) \ \& \ B \in L(R^{n-p},R^{n-p}).$$

Again, with respect to this decomposition, we have:

$$M=\begin{pmatrix} I_p & X \\ O & Y \end{pmatrix}, \text{where } Y \in L(R^{n-p},R^{n-p}) \ \& \ X \in L(R^p,R^{n-p}).$$

This follows since G(x) is a fixed set for f & so $T_xG(x)$ is left fixed by T_xf.

Thus condition 1 above may be written:

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}\begin{pmatrix} I_p & X \\ O & Y \end{pmatrix}\begin{pmatrix} A^{-1} & O \\ O & B^{-1} \end{pmatrix}=\begin{pmatrix} I_p & X \\ O & Y \end{pmatrix}$$

Which is clearly equivalent to:

$$BYB^{-1}=Y \quad \ldots \quad L.$$

$$AXB^{-1}=X \quad \ldots \quad M.$$

Now set $X_a=aX, Y_a=aY$, for $a \in R$. Then it is clear that X_a & Y_a satisfy L. & M.

Now either Y has no eigenvalues modulus 1, in which case f is 1-generic on G(x) & there is nothing to prove, or not. If not, $\exists h>0$, s.t. if $0<|1-a|\leqslant h$, then Y_a has no eigenvalues modulus 1.

We define for $\mu \in [0,1]$, $f_\mu^*: D \subset R^p \times R^{n-p} \longrightarrow R^n$ as follows:

$$f^*_\mu(y)=\begin{pmatrix} I_p & X \\ 0 & Y \end{pmatrix}y+\mu q(y)\begin{pmatrix} 0 & X \\ 0 & Y \end{pmatrix}y+G(y), \text{where } y\in D \text{ \& } q \text{ is a}$$

C^{∞} function from R^n to R s.t.:

1. $q=h$ on some disc nbd. J of $\{0\}$ in R^n.

2. $0\leq q\leq h$.

3. $q=0$ outside of some disc nbd. P of $\{0\}$ in R^n, where P is chosen s.t.:

$$\forall\mu\in[0,1] \quad \overline{f^*_\mu(P)\cap P}\subset D$$

4. q is G_x-invariant.

That functions satisfying 1,2 & 3 exist is standard. Let q' be such a function, then set $q=Av(q')$, then q satisfies the reqmired conditions.

We note, in particular, that $\|q\|_r<\infty$.

Now f^*_μ is G_x-invariant, since q, G & M are & G_x acts linearly on R^n.

Consider $f^*_\mu : S_x\cap D\longrightarrow R^n$. We note that $S_x\cap D$ is a slice at x. Working in M we define:

$$\bar{f}^*_\mu : G(S_x\cap D)\longrightarrow M \text{ by:}$$

$$\bar{f}^*_\mu(gy)=gf^*_\mu(y), g\in G, y\in S_x\cap D.$$

Then we assert that $\bar{f}^*_\mu$ is a well-defined G-invariant C^r function which equals f on some collar nbd. of $G(S_x\cap D)$.

The second statement is immediate, by the construction of f^*, i.e. $f^*_\mu=f^*$ in some collar nbd. of D (i.e. the boundary of D).

We first show $\bar{f}^*_\mu$ is well defined.

Suppose $gs=ht$, s & $t\in(S_x\cap D)g$ & $h\in G$. Then we have to prove: $gf^*(s)=hf^*(t)$

i.e. that: $h^{-1}gf_\mu^*(s)=f^*(t)$

i.e. that: $f^*(h^{-1}gs)=f_\mu^*(t)$, using the G_x-invariance of f^*.

i.e. that: $f_\mu^*(h^{-1}(ht))=f_\mu^*(t)$.

Thus $\bar{f}_\mu^*$ is well defined, the G-invariance follows from the definition & C^r is obvious.

Since $\bar{f}_\mu^*=f$ on a collar of the boundary of $G(S_x\cap D)$ $\bar{f}_\mu^*$ extends to f_μ' on M, with $f_\mu'=f$ outside of $G(S_x\cap D)$.

Now f_1' may not be a diffeomorphism, if it is not we note that as $\mu\longrightarrow 0$, $f_\mu'\longrightarrow f$ in the C^r topology. This is a consequence of the fact that $\|\mu q\|_r\longrightarrow 0$ as $\mu\longrightarrow 0$.

Thus using the openness of the set of equivariant diffeomorphisms in the C^r topology, there exists c, $0<c\leq 1$, s.t. f_a' is a diffeomorphism $0<a\leq c$. Thus we may make f_a' arbitrarily C^r close to f & by construction f_a' is 1-generic on $G(x)$, $0<a\leq c$. Locally we have:

$$Df_a'(x)=\begin{pmatrix} I_p & aX \\ 0 & aY \end{pmatrix},$$ where the splitting is given by $T_xG(x)\times S_x$.

Remark:

In the proof of the above we have shown that our perturbation $f_a'=f$ outside of some G-tubular nbd. of $G(x)$, i.e. if type$(y)\not>$type(x), $f_a'(y)=f(y)$

Lemma 5

Let $s \in C^r_G(TM)$ & let $G(x)$ be a singular set for s, then there exists an $s' \in C^r_G(TM)$, arbitrarily C^r close to s, s.t. $G(x)$ is a singular set for s' & s' is 1-generic on $G(x)$.

Proof

The proof is similar, though more straightforward, to that of Lemma 4.

As in the proof of Lemma 4, we take the G_x-normal bundle $T_xM = R^n$ to x, x regarded as a fixed point of the G_x action on M. We take $T_xG(x) = R^p \times \{0\}$ & $S_x = \{0\} \times R^{n-p}$.

Now since G_x acts linearly on R^n, if $A \in G_x$, then $DA = A$. Thus noting that $TR^n = R^n \times R^n$, the induced G_x-action on TR^n is of the form:

$$A: R^n \times R^n \longrightarrow R^n \times R^n; (x,y) \longmapsto (Ax, Ay), A \in G_x.$$

We have a local representitive s^* of s, s.t.:

$$s^*: R^p \times R^{n-p} \longrightarrow T(R^p \times R^{n-p}) = (R^p \times R^{n-p}) \times (R^p \times R^{n-p})$$

Now $s^*(t) = (t; S^*(t)), t \in R^n$.

Using Taylor's expansion:

$$S^*(t) = Mt + G(t), \text{where } M \in L(R^n, R^n) \text{ \& } G(t) = o(t).$$

Noting the above remarks on the G_x-action on TR^n, we see that, as in Lemma 4, M & G are equivariant, i.e.:

$$AMA^{-1} = M, (A \in G_x) \ldots\ldots\ldots\ldots 1.$$

Restricting attention to the second factor of TR^n, we see that any $g \in G_x$ is of the form:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \text{where } A \in L(R^p, R^p) \text{ etc.}$$

Also M is of the form:

$$M=\begin{pmatrix} 0 & X \\ 0 & Y \end{pmatrix}, \text{where } X\in L(R^{n-p},R^{p}) \ \& \ Y\in L(R^{n-p},R^{n-p})$$

Thus as equivalent conditions to 1 we have:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & X \\ 0 & Y \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}=\begin{pmatrix} 0 & X \\ 0 & Y \end{pmatrix} \text{i.e.:}$$

$$BYB^{-1}=Y \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ L'$$

$$AXB^{-1}=X \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ M'$$

Given $a\in R$, let us define $Y^{a}=Y-aI$. Then X & Y^{a} satisfy L' & M'. If Y has no eigenvalues real part zero, s is 1-generic on $G(x)$ & there is nothing to prove. If not, $h>0$, s.t. if $0<a\leq h$, Y^{a} has no eigenvalues real part zero

Define:

$S^{*}:R^{n}\longrightarrow R^{n}$ as follows:

$$S^{*}_{\zeta}(y)=\begin{pmatrix} 0 & X \\ 0 & Y \end{pmatrix}y-\zeta q(y)\begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix}y+G(y), \text{where } q \text{ is the}$$

function defined in Lemma 4, $\zeta\in[0,1]$.

S^{*}_{ζ} is clearly G_{x}-invariant, since q & G & M are, & $S^{*}_{\zeta}=S^{*}$ outside some disc nbd. of the origin in R^{n}.

Thus we may define a new vectorfield s^{*}_{ζ} on $R^{n}\subset M$, whose principal part is S^{*}_{ζ}.

As in Lemma 4, we may extend s^{*}_{ζ} to $s^{!}_{\zeta}$ defined on M, by restriction to S_{x} & then extending.

As $\zeta\longrightarrow 0$, $s^{*}_{\zeta}\longrightarrow s$ in C^{r}-topology & so we may arbitrarily C^{r} approximate s by $s^{!}_{\zeta}$.

$s^{!}_{\zeta}$ is 1-generic on $G(x)$, in fact, locally we have:

$$Ds^{!}_{\zeta}(x)=\begin{pmatrix} 0 & X \\ 0 & Y-\zeta I \end{pmatrix}$$

Finally, we again remark that $s'_\zeta = s$ outside of some G-tubular nbd. of $G(x)$.

6. <u>1-generic is a generic property</u>

Let $\mathfrak{A}^1_G(M,r)=\{f\in\mathfrak{Diff}^r_G(M):f$ is 1-generic$\}$.

& let $\mathfrak{X}^1_G(M,r)=\{X\in C^r_G(TM):X$ is 1-generic$\}$.

We assume, as usual, that M is compact.

Clearly if $f\in\mathfrak{A}^1_G(M,r)$ then $f\bar{\pitchfork}_G\Delta(M)$ & if $X\in\mathfrak{X}^1_G(M,r)$ then $X\bar{\pitchfork}_G(TM)_0$.

If E is a G-fiber bundle over M, then:

$ev:C^r_G(E)\times M\longrightarrow J^1(E)$ is continuous $(r\geq 1)$, so that, using Theorem 7, it is easy to see that both sets of 1-generic maps defined above are open. Thus 1-generic is an open property for these sets.

The aim of this section is to prove that:

1. $\mathfrak{A}^1_G(M,r)\subset\mathfrak{Diff}^r_G(M)$ as a dense set.
2. $\mathfrak{X}^1_G(M,r)\subset C^r_G(TM)$ as a dense set.

We will give a detailed proof for the diffeomorphism case only, the result for $C^r_G(TM)$ is proved similarly.

Recall that we have a partial order   on M, defined by the orbit decomposition $M=M_1\cup\ldots\cup M_N$.

Let $a\in\{1,\ldots,N\}$ be s.t. $j\leq a$ implies $j=a$. Call such a <u>minimal</u>.

<u>Proposition 12</u>

If a is minimal, M_a ia a <u>closed</u> submanifold of M.

<u>Proof</u>

M_a is certainly a submanifold of M. If M_a is not closed $\exists$ a sequence of points x_n of M_a, s.t. $x_n\longrightarrow x\notin M_a$. But, using a slice at x, we see that type$(x)<a$. Contradiction by the

minimality of a.

Let $i\in\{1,\dots,N\}$. Define, for $x\in M_i$, $M_i^x=\{z\in M_i : G_z=G_x\}$. Then $M_i^x\subset M_i$ as a submanifold & $G(M_i^x)=M_i$ (See Appendix 1). Note that if i is minimal M_i^x is closed.

With the natural induced G-action we define $(M \times M)_i^x$ in the obvious way.

It is clear that $(M \times M)_i^x \supset M_i^x \times M_i^x$, in fact we have:

<u>Proposition 13</u>

$M_i^x \times M_i^x$ is an isolated subset of $(M \times M)_i^x$.

If i is minimal, $M_i^x \times M_i^x=(M \times M)_i^x$.

<u>Proof</u>

Suppose $(z,y)\in M_i^x \times M_i^x$. Then $G_z=G_y=G_x$.

$\exists$ nbds U_z & V_y of z & y respectively s.t. if $s\in U_z$ or V_y then G_s is conjugate to a subgroup of G_x (See section 1, Part 2). Thus if $(s,t)\in U_z \times V_y$ then $G_{(s,t)}$ is conjugate to a subgroup of G_x & is equal to G_x iff $G_s=G_t=G_x$ –we note here that $(z,y)\in(M \times M)_i^x$ iff $G_z\cap G_y=G_x$, this also implies that the second statement of the proposition is true.

Consequently $(U_z \times V_y)\cap(M \times M)_i^x\subset M_i^x \times M_i^x$.

Now define:

$$W=\bigcup_{\{(z,y)\in M_i^x \times M_i^x\}}(U_z \times V_y).$$

This is clearly a nbd. of $M_i^x \times M_i^x$ disjoint from $(M \times M)_i^x-M_i^x \times M_i^x$.

Now $M_i^x=\bigcup_{j\in J_i} M_i^j$, where the M_i^j are the connected submanifold components of M_i^j, J_i is finite. We set $\dim(M_i^j)=q_i^j$.

Now we assert that $\Delta(M)_i^x = \bigcup_{\{(m,m)\in\Delta(J_i)\}} \Delta(M_i^j)$. This is obvious, since the left hand side is certainly contained in $M_i^x \times M_i^x$.

Thus $\Delta(M)_i^x \subset M_i^x \times M_i^x = \bigcup_{\{(m,l)\in J_i \times J_i\}} M_i^l \times M_i^m$ & each connected component $\Delta(M_i^j)$ of $\Delta(M)_i^x$ is contained in the connected component $M_i^j \times M_i^j$ of $(M \times M)_i^x$ as a submanifold of codimension q_i^j, by Proposition 13.

Define:

$$E_i^j = \{f \in C_G^r(M, M \times M) : (f|M_i^j) \pitchfork \Delta(M_i^j) \text{ in } (M \times M)_i^x\}.$$

Lemma 3 gives us immediately that E_i^j is a dense subset of $C_G^r(M, M \times M)$.

Now if a is minimal, M_a^j is closed & so compact, therefore E_a^j is open (openness of transversal intersection).

Now, by standard properties of transversality, if $f \in E_a^j$ then $(f|M_a^j)^{-1}\Delta(M_a^j)$ is a submanifold of M_a^j of codimension 0. Thus, since M_a^j is compact $(f|M_a^j)^{-1}\Delta(M_a^j)$ is a finite set of points, say $\{x_k\}_{k\in Q_a^j}$.

Now each $G(x_k)$, $k \in Q_a^j$, is a fixed set for f on $G(M_a^j)$. We may construct a set of mutually disjoint G-tubular nbds U_k of $G(x_k)$. Lemma 4 implies that we may approximate f by f' s.t. f' is 1-generic on $G(x_k)$ & f=f' outside of $M - \bigcup_{k\in Q_a^j} U_k$. Since E_a^j is open we may suppose that no new fixed sets are introduced on M_a^j—here we use Theorem 7 on $G(M_a^j)$.

Thus letting $H_a^j = \{f \in C_G^r(M, M \times M) : f|G(M_a^j) \text{ is 1-generic}\}$, we see that H_a^j is an open & dense subset of $C_G^r(M, M \times M)$.

Since J_a is finite:

$$H_a = \bigcap_{j \in J_a} H_a^j \text{ is also an open \& dense subset of}$$
$C^r_G(M, M \times M)$.

Now the relation $<$ enables us to define the graph $\mathfrak{G}(\{1,..,N\}) = \mathfrak{G}_N$ of (M,G). Vertices of this graph are points $i \in \{1,..,N\}$ & ij is a directed edge iff:

1. $i<j$ & $\nexists k$ s.t. $i<k<j$ &
2. $\exists x$, with type$(x)=i$, s.t. $x \in \partial M_j$.

Here we are directing the edges of our graph $j \longrightarrow i$.

If K is a finite graph, define vert(K) = set of vertices of K.

We say H is a subgraph of $\mathfrak{G}_N$ iff vert$(H) \subset \{1,..,N\}$ & the set of edges of H is defined by the relation $<$ .i.e. $ij \in H$ iff $ij \in \mathfrak{G}_N$ & i & $j \in$ vert(H).

If $J \subset \{1,..,N\}$ we define $\mathfrak{G}(J)$ as the graph generated on this subset by $<$. $\mathfrak{G}(J)$ is a subgraph of $\mathfrak{G}_N$.

We assert that given $\mathfrak{G}_N$, we may construct a sequence of graphs $(\mathfrak{G}_k)_{0<k\leq N}$ s.t.:

1. $\mathfrak{G}_k$ is a subgraph of $\mathfrak{G}_N$ with k vertices.
2. $\mathfrak{G}_k \subset \mathfrak{G}_{k+1}$ as a subgraph, $1 \leq k \leq N-1$.
3. $\mathfrak{G}_1$ is a vertex corresponding to a minimal point.

We note that 2. implies that if $j \in \mathfrak{G}_{k+1}$, $j \notin \mathfrak{G}_k$ then if $i<j$, $i \in \mathfrak{G}_k$.

First we define the idea of a maximal element.
If $J \subset \{1,..,N\}$, say t is a maximal element of $\mathfrak{G}(J)$ if for $k \geq t$, $k \in \mathfrak{G}(J)$ then $k=t$.

Define $\mathfrak{G}_{N-1} = \mathfrak{G}(\{1,..,N\} - \{t_1\})$, where t_1 is a maximal element of $\mathfrak{G}_N$.

$\mathfrak{G}_{N-1}$ is obviously a subgraph of $\mathfrak{G}_N$ with one less vertex.

We proceed inductively:Suppose $\mathfrak{G}_{N-r}$ has been defined, then,if $r<N-1$,we set

$$\mathfrak{G}_{N-r-1}=\mathfrak{G}(\mathrm{vert}(\mathfrak{G}_{N-r})-\{t_{r+1}\}),$$ where t_{r+1} is a maximal element of $\mathfrak{G}_{N-r}$.

If $N-r=1$,then $\mathrm{vert}(\mathfrak{G}_1)=$a minimal point by construction.

The checking that 1,2, & 3 are true is trivial.

Let $P_k=\mathrm{vert}(\mathfrak{G}_k)\subset\{1,\ldots,N\}$.

Corresponding to P_k we have a subset $I_k=\bigcup_{j\in P_k} M_j$ of M. We note that I_k is closed,$1\leq k\leq N$.

Define:

$$H_i^j=\{f\in C_G^r(M,M\times M):f|(M_i^j\cup I_{i-1}) \text{ is 1-generic}\}.$$
$$H_i=\bigcap_{j\in J_i} H_i^j=\{f\in C_G^r(M,M\times M):f|I_i \text{ is 1-generic}\}.$$

We have already proved that,if a is minimal,H_a is an open & dense subset of $C_G^r(M,M\times M)$.Suppose that H_i is open & dense,$i\leq k$.We will prove that H_{k+1} is open & dense & hence,by induction, $\mathfrak{A}_G^1(M,r)=H_N$ will be open & dense.

Now $E_i^j=\{f\in C_G^r(M,M\times M):(f|M_i^j)\,\overline{\pitchfork}\,\Delta(M_i^j)$ in $(M\times M)_i^x$ is a dense subset of $C_G^r(M,M\times M)$.

Set $F_{k+1}^j=H_k\cap E_{k+1}^j$,$j\in J_k$. Let $f\in F_{k+1}^j$.

Since H_k is open & dense,by the inductive hypothesis, F_{k+1}^j is a dense subset of $C_G^r(M,M\times M)$.

Consider $(f|M_{k+1}^j)^{-1}\Delta(M_{k+1}^j)$.This is a submanifold of codimension 0.

i.e. $(f|M^j_{k+1})^{-1}\Delta(M^j_{k+1})=\{x_l\}_{l\in Q^j_{k+1}}$. We assert that Q^j_{k+1} is finite. For, if not, $\exists$ a convergent sequence of distinct points $x_m, m\in Q^j_{k+1}$, s.t. $x_m \longrightarrow x$. Clearly $fx=x$. Now $x\notin M^j_{k+1}$, since $(f|M^j_{k+1})\bar{\pitchfork}_x\Delta(M^j_{k+1})$ & therefore x is an isolated fixed point. Now $x\in M_b$ for some b, clearly type(x)<k+1, since $I_k \cup M_{k+1}$ is closed. But $f|M_b$ is 1-generic by the inductive hypothesis & therefore G(x) is an isolated fixed set of f. Contradiction, therefore Q^j_{k+1} is finite.

Let us take a cover of the $G(x_l)$ by mutually disjoint G-tubular nbd. pairs (U_l, V_l) s.t. $U_l \supset \bar{V}_l \supset G(x_l), l\in Q^j_{k+1}$. Noting that $U_l \cap I_k = \emptyset$, we have, by Lemma 4, arbitrarily good C^r approximations f' of f s.t. f'=f on $I_k \cup (M^j_{k+1} - \bigcup_{l\in Q^j_{k+1}} V_l)$ & f' is 1-generic on $G(x_l), l\in Q^j_{k+1}$.

Now since $\bar{V}$ is compact we may assume by the transversality condition for F^j_{k+1} that $G(x_l)$ is the only fixed set for f' in V_l. Thus $f'\in H^j_{k+1}$ & H^j_{k+1} is dense in H_k.

We must also show that H^j_{k+1} is open. To do this we make use of the fact that $I_k \cup G(M^j_{k+1})$ is closed.

First note that if $f\in H^j_{k+1}, (f|I_k \cup M^j_{k+1})^{-1}\Delta(M)$ consists of a finite set of isolated G-orbits, $G(x_l), x_l \in I_k \cup M^j_{k+1}$, l P.

We take a set of mutually disjoint G-tubular nbd. pairs (U_l, V_l) covering these orbits s.t. $U_l \supset \bar{V}_l$. Now, as in the proof of Theorem 7, we may find a nbd. N^1 of f in $C^r_G(M, M \times M)$ s.t. if $g\in N^1$:

$V_l \cap (g|I_k \cup M^j_{k+1})^{-1}\Delta(M)$ contains at most one G-orbit of the inverse set.

The condition that $(g|(I_k \cup M^j_{k+1}))^{-1}\Delta(M)\cap(M-\bigcup_{l\in P} V_l)=\emptyset$

is an open condition, since $I_k \cup M^j_{k+1}$ is closed. Thus $\exists$ nbd. N^2 of f in $C^r_G(M, M \times M)$ s.t. if $g\in N^2$ the above relation is satisfied.

Let $N=N^1\cap N^2\cap H_k$.

Now, using the fact that $ev: C^r_G(E) \times M \longrightarrow J^1(E)$ is continuous, we may insist that $N\subset H^j_{k+1}$.

Therefore $H_{k+1}=\bigcap_{j\in J_{k+1}} H^j_{k+1}$ is open & dense in H_k & so

in $C^r_G(M, M \times M)$, completing our induction.

Now $\mathfrak{Diff}^r_G(M)$ is an open submanifold of $\mathfrak{C}^r_G(M,M)$, therefore we may state:

<u>Theorem 8</u>

$\mathfrak{A}^1_G(M,r)\subset\mathfrak{Diff}^r_G(M)$ as an open & dense set.

In a similar manner one may also prove:

<u>Theorem 9</u>

$\mathfrak{N}^1_G(M,r)\subset C^r_G(TM)$ as an open & dense set.

Noting that if $f\in\mathfrak{Diff}^r_G(M)$ is 1-generic then $f\pitchfork_G\Delta(M)$ we have, using Theorem 7 & the continuity of the evaluation map, an appropriate Isotopy theorem for 1-generic maps. Similarly for 1-generic vectorfields.

Now one feature of 1-generic maps is that under arbitrarily small C^r perturbations fixed sets may vanish, see Example 1, we now show that generically we may suppose that this does not happen.

For suppose $f \in E_i^j = \{g \in C_G^r(M, M \times M) : (f|M_i^j) \pitchfork \Delta(M_i^j)$ in $(M \times M)_i^x\}$. Then we know that $(f|M_i^j)^{-1}\Delta(M_i^j)$ is a collection of isolated points $\{x_k\}_{k \in Q}$, & $G(x_k)$ is a fixed set for f. Now we know that $G_x(x_k) = x_k$, by choice of M_i^j. Suppose that $\dim(N(G_x)) > \dim G_x$, then $N(G_x)x_k$ will be a submanifold of dimension at least 1, containing x_k. But $N(G_x)x_k \subset M_i^j$ & thus $N(G_x)x_k \subset (f|M_i^j)^{-1}\Delta(M_i^j)$. Contradiction. Therefore, if $(f|M_i^j)^{-1}\Delta(M_i^j) \neq \emptyset$ we always have $\dim(N(G_x)) = \dim G_x$. See also the remark following Lemma 3.

Definition 16

Say $f \in \mathfrak{Diff}_G^r(M)$ is 1*-generic if f is 1-generic & f has no fixed sets $G(x)$, where $\dim(N(G_x)) > \dim G_x$.

Denote the set of 1*-generic maps by $\mathfrak{A}_G^{1*}(M,r)$

The above remark, together with an examination of the proof of Theorem 8, proves that:

Theorem 10

$\mathfrak{A}_G^{1*}(M,r) \subset \mathfrak{Diff}_G^r(M)$, as an open & dense set.

Similarly, for the vectorfield case we have:

Definition 17

Say $s \in C_G^r(TM)$ is 1*-generic iff s is 1-generic & s has no singular sets $G(x)$, where $\dim(N(G_x)) > \dim G_x$.

Denote the set of 1*-generic maps by $\mathfrak{X}_G^{1*}(M,r)$.

As in the diffeomorphism case we have:

Theorem 11

$\mathfrak{X}_G^{1*}(M,r) \subset C_G^r(TM)$ as an open & dense set.

Again, it is easy to see that if $f \in \mathfrak{Diff}_G^r(M)$ is 1*-generic then, under perturbation, no fixed sets vanish-this is a consequence of the $\pitchfork$ in the definition of E_i^j. Similarly for the vectorfield case. Thus we have:

Theorem 12

If $f\in\mathfrak{Diff}_G^r(M)$ is 1*-generic, then $\exists$nbd. N of f in C^r topology, s.t. if $g\in N$, $\exists$an equivariant C^∞ isotopy K_g between the fixed sets of f & g.

Theorem 13

If $s\in C_G^r(TM)$ is 1*-generic, then $\exists$nbd. N of s in C^r topology, s.t. if $g\in N$, $\exists$an equivariant C^∞ isotopy K_g between the singular sets of s & g.

We conclude this section with a definition:

Definition 18

We say $f\in\mathfrak{Diff}_G^r(M)$ is 2-generic iff $f^n\in\mathfrak{A}_G^1(M,r)$, for all non-zero integers n.

We say f is 2*-generic iff f^n is 1* generic, for all non-zero integers n.

We denote the respective sets of 2-generic (2*-generic) diffeomorphisms by $\mathfrak{A}_G^2(M,r)$ ($\mathfrak{A}_G^{2*}(M,r)$).

7. Closed orbits of Equivariant vectorfields

This section is devoted to a few preliminary results concerning closed orbits of Equivariant vectorfields & the flow structure on G-orbits left invariant by the flow.

First, for completeness, we state & prove a result, the first part of which we have used already (see, for example, Wasserman 1).

Proposition 14

1. If $s \in C^r_G(TM)$ & $F^{(s)}: M \times R \longrightarrow M$ is the flow of s, then $F^{(s)}$ is equivariant, i.e. $F(gx,t)=gF(x,t)$ $(g \in G)$.

2. The isotropy group $G_{\cdot}$ is constant on integral curves of $F^{(s)}$.

Proof

1. Let $F_x(t)$ be the integral curve of s through x, then:

$s \cdot F_x(t) = DF_x(t,1)$. Using the equivariance of s we have: $s(gF_x(t)) = Dg(s \cdot F_x(t)) = DgDF_x(t,1)$

$= D(gF_x(t,1))$.

i.e. gF_x is the integral curve of s through gx. The uniqueness of solutions gives the result.

2. Suppose $g \in G_x$, then $gF_x(t) = F_{gx}(t) = F_x(t)$. Thus $G_{F_x(t)} \supset G_x$. Converse is also true, giving result.

Definition 19

If q is a closed orbit of $s \in C^r_G(TM)$, we set

$G_q = \{g \in G : g(q) = q\}$. Call G_q 'The stabiliser group of q'.

Remark: If q is a closed orbit of s, so is gq, $g \in G$ & it is easy to see that $G_{gq} = gG_qg^{-1}$.

Proposition 15

Let $x \varepsilon q$, then:

1. G_q is a closed subgroup of G.
2. $G_x \subset G_q$ as a closed normal subgroup.

Proof

1. Obvious: Since G & q ($\cong S^1$) are compact.
2. Clearly $G_x \subset G_q$ giving the first part of 2.

Now $G_{gx} = gG_x g^{-1}$, but if $g \varepsilon G_q$, $G_{gx} = G_x$, by Proposition 15, Therefore $G_x \triangleleft G_q$.

Proposition 15 tells us that G_q/G_x is a Lie group, we have in fact:

Proposition 16

$G_q/G_x \cong C^r$ or S^1, where C^r is the cyclic group of order $r, r=1,2,\ldots$.

Proof

Let x be some point of q, fixed once & for all.

Let T be the prime period of q, then any point $m \varepsilon q$ is uniquely representable as $F_t(x), 0 \leqslant t < T$.

Let us orient q in the direction of positive with t increasing.

For all $t \varepsilon [0,T)$, either there exists $g_t \varepsilon G_q$ s.t. $g_t x = F_t(x)$, or not.

If not, there exists a minimum strictly positive value of t, say P, s.t. there exists $g_p \varepsilon G_q$ with $g_p(x) = F_P(x)$, but $\nexists g \varepsilon G_q$ s.t. $gx = F_t(x)$ with $0 < t < P$. We consider the set of points:

$$g_p(x) = F_P(x), g_p^2(x) = F_{2P}(x), \text{etc.}$$

It is an easy exercise to check that we have the following sequence of isomorphisms:

$$\xrightarrow{g_p^{\approx}}\left[g_p^{-1}(x),x\right]\xrightarrow{g_p^{\approx}}\left[x,g_p(x)\right]\xrightarrow{g_p^{\approx}}\left[g_p(x),g_p^2(x)\right]\xrightarrow{g_p^{\approx}}\cdots\cdots$$

Noting that $g_p^r(x)=F_{rP}(x)$, $\exists$ some least positive integer $n\geq 2$, s.t. $g_p^{n-1}(x)=x$, or lies to the left of x, & $g_p^n(x)$ lies strictly to the right of x.

Now $g_p^{n-1}(x)=F_{-r}(x)$, for some $r, 0\leq r<P$, & thus $g_p^n(x)=F_{P-r}(x)$. But, by our choice of g_p, r must be zero, since otherwise $g_p^n(x)=F_{P-r}(x)\varepsilon(x,g_p(x))$.

Suppose $\exists h\varepsilon G_q$ s.t. $hx\varepsilon(g_p^r(x),g_p^{r+1}(x))$. Using the above sequence of isomorphisms $g_p^{-r}h(x)\varepsilon(x,g_p(x))$. Contradiction, by choice of g_p.

Consider the set $\{g_p^r : r=0,1,..,n-1\}$. We may represent each element of G_q/G_x as a coset $\{hG_x\}$, $h\varepsilon G_q$. We assert that the set of cosets $\{g_p^r G_x\}_{0\leq r\leq n-1}$ is isomorphic to G_q/G_x. This is obvious, since $\{hG_x\}$ & $\{fG_x\}$ represent the same element of G_q/G_x iff $f^{-1}h\varepsilon G_x$ iff $fx=gx$. q.e.d.

Suppose the first possibility occurs.

Consider $G_q/G_x=\{gG_x\}_{g\varepsilon G_q}$.

$gG_x=rG_x$ iff $rx=gx$. Therefore each point $z\varepsilon q$ defines a unique coset of G_q/G_x & conversely. (We have implicitly used the compactness of G_q here to show $G_q(x)=q$)

Thus G_q/G_x is a 1-dimensional connected Lie group & so isomorphic to S^1-in fact naturally to q with the multiplication defined by the flow mod T.

<u>Proposition 17</u>

If $G_q/G_x=S^1$ then there exists $S^1\subset G_q$ s.t. $G_q=S^1.G_x$.

The product is not generally semi-direct (unless $S^1 \cap G_x = e$)

Proof

Let g^x & g^q denote the Lie algebras of G_x & G_q respectively. Then $g^x \subset g^q$ as a vector subspace.

We have 1-parameter subgroups of G_q, $\{a_t^Y\}$, defined for each $Y \in g^q$ (see, for example, Kobayashi & Nomizu 1, page 38ff) Denote the set of these subgroups by $\{a_t^Y\}_{Y \in g^q}$. Now $X \in g^q$, s.t. $X \notin g^x$. Let us consider V = orbit of a_t^X. There are two possibilities:

1. V is closed in G_q. This implies that V is an S^1, i.e. a closed orbit.

2. If V is not closed, $\bar{V}$ is, & $\bar{V}$ is a closed Abelian connected Lie subgroup of G_q, i.e. $\bar{V}$ is a torus. We may clearly choose an S^1 subgroup of $\bar{V}$ s.t. S^1 is not wholly contained in G_x, using the fact that $X \notin g^x$ is an open relation.

We now show that the S^1 constructed above satisfies the conditions of the Proposition.

We first note that $S^1(x) = q$. For, if not, $\exists P \in (0,T)$ s.t. for all $g \in S^1$ $gx = F_t(x)$, for some t s.t. $0 \leq t \leq P$ & P is the smallest positive number satisfying this property, P may be supposed non-zero since $S^1 \not\subset G_x$ (P, of course, may be strictly negative, but we assume P positive w.l.o.g., i.e. if P is negative set $X' = -X$ & use X').

Therefore $\exists h \in S^1$, s.t. $hx = F_P(x)$ - since S^1 is compact.

But let $gx = F_e(x)$, where e is s.t. $P + e < T$, $g \in S^1$.

Then $gh(x) = F_{P+e}(x)$, contradiction by the definition of P. Therefore $S^1(x) = q$.

Now we assert that any $g\epsilon G_q$ may be written $g=rh, r\epsilon G_x$, $h\epsilon S^1$. For, given $g\epsilon G_q$, choose $h\epsilon S^1$, s.t. $hx=gx$. Then, since $\{hG_x\}$ is a coset in G_q/G_x, there exists an $r\epsilon G_x$ s.t. $g=rh$. Thus $G_q=G_x\cdot S^1$.

We note also that $G_q=S^1.G_x$. Since, if $s\epsilon S^1, h\epsilon G_x$, then $g=s(s^{-1}hs)$ & $s^{-1}hs\epsilon G_x$, using $G_x \triangleleft G_q$.

<u>Remark</u>: We note that $q\subset G(x)$ iff $G_q/G_x \cong S^1$.

Since G_q may be considered as a group action on M, by restriction, & since q is the G_q orbit of $x\epsilon q$, we have at each point x of q a G_q-slice, which we will denote by Q_x.

<u>Definition 20</u>

Let $G_q=S^1.G_x$.

With the above notation, let $z\epsilon Q_x$, then let us define p(z), 'the period of z w.r.t. S^1', to be the number of points of intersection of $S^1(z)$ with Q_x.

<u>Proposition 18</u>

1. p(z) is finite ($z\epsilon Q_x$) & in fact we have:
2. There exists a least positive integer P, s.t. $p(z) \mid P, z\epsilon Q_x$.

<u>Proof</u>

1. Since S^1 is a compact Lie group & since $G_q(Q_x)$ is a G_q-invariant tubular nbd. of q, $S^1(z)$ is a submanifold of $G_q(Q_x)$ of dimension 1. Thus, since $S^1(z) \bar{\pitchfork} Q_x$, $S^1(z)\cap Q_x$ is a finite set of points.

2. Let us assign to S^1 an orientation, corresponding to that of q.

Now fixing attention to $id\epsilon S^1$, $id(x)=x$. $\exists$a nbd. V of the in S^1 s.t. if $g\epsilon V-\{id\}$, then $gx\neq x$. Thus $\exists$a least $h\epsilon S^1$, strictly to the right of the id s.t. $hx=x$ (h may equal the identity) Now, as in the proof of Proposition 16, we may show that $\exists P^*>0$, s.t. $S^1(x)\cap\{x\}=\{x,hx,h^2x,..,h^{P^*-1}x\}$.

We assert that $p(z)\leq P^*$ $(z\epsilon Q_x)$

For if $z\epsilon Q_x$ then $fz=z$ implies $f\epsilon G_x$, since Q_x is a slice for G_q at x. Now if, in addition, $f\epsilon S^1$, then $f=h^r$ for some r, $0\leq r\leq P^*-1$ & therefore $p(z)\leq P^*$.

Finally, we choose the smallest number P s.t. $p(z)|P$, $z\epsilon Q_x$ & P positive.

Definition 21

Say P is the period of S^1 w.r.t. q.

Remark:

Let us take a G-slice S_x at $x\epsilon q$. We assert that $S^1(z)$ meets S_x in precisely $p(z)$ points, where $p(z)$ is the period of z, if $p(z)$ is defined.

For, let h be defined as in Proposition 18, then $hx=x$ & so $hS_x=S_x$. Thus $hz\epsilon S_x$. Further $\nexists$any h' lying between id & h s.t. $h'z\epsilon S_x$, since by construction of h, $h'\notin G_x$. Consequently $S^1(z)$ meets S_x in precisely $p(z)$ points.

The remainder of this section is devoted to a study of the orbit structure of equivariant vectorfields defined on the homogeneous space G/H, where $H\subset G$ as a (closed) Lie subgroup.

We are thinking here particularly of the case where $H=G_x$ & $G/H\cong G(x)$ & $G(x)$ is left invariant by the flow, for example, when $q\subset G(x)$ or x is a fixed point.

Let H be a closed (Lie) subgroup of G. We consider the set of G-invariant sections of $T(G/H)$. Since G is transitive on G/H, it is clear that if s is a section of $T(G/H)$, then s is defined by its value at any point of G/H. Further (looking at local cross-sections of G_x, $x\in G/H$) it is easy to see that all equivariant sections are C^{∞}. Thus we will consider $C^{\infty}_G(T(G/H))$.

Let $s\in C^{\infty}_G(T(G/H))$, there are three possibilities:

1. s has a fixed point on G/H.
2. s has a closed orbit on G/H.
3. Neither 1. nor 2. occurs.

We examine these three cases in more detail:

1. If $s(x)=0_x$, for some $x\in G/H$, then $s=0$ on G/H. In particular, therefore, if $\chi(G/H)\neq 0$, there are no non-trivial equivariant vectorfields on G/H.

The converse is, however, false; e.g. $SO(4)/SO(3)\cong S^3$.

2. We have that G/H is foliated by circles, see also case 3.
3. Let $x\in G/H$ & let $O(x)$ denote the s-orbit through x.

Define $G^*_x=\{g\in G: gx=F^s_t(x)$, for some $t\}$. F^s is the flow of s.

<u>Proposition 19</u>

G^*_x is a subgroup of G.

<u>Proof</u>: Trivial.

Clearly $O(x)=G^*_x(x)$. Now $O(x)$ & G^*_x are not closed, for otherwise $O(x)$ would be a point (case 1.) or an S^1 (case 2.).

Proposition 20

1. $G_x \triangleleft G_x^*$.

2. G_x^*/G_x is a commutative group.

Proof

1. Let $g \varepsilon G_x^*$ & $h \varepsilon G_x$. We consider $ghg^{-1}(x)$. Now $g(x)=F_t(x)$ for some t, & $g^{-1}(x)=F_{-t}(x)$. Further the isotropy group is constant on $O(x)$, therefore:

$ghg^{-1}(x)=ghF_t(x)=gF_t(x)=F_t(gx)=F_t(F_{-t}(x))=x$, therefore we have $ghg^{-1} \varepsilon G_x$.

2. If $F_t(x)=gx$, $F_{t'}(x)=g'x$, where g & $g' \varepsilon G_x^*$ then:

$$F_{t+t'}(x)=g'F_t(x)=g'gx.$$

$$F_{t'+t}(x)=gF_{t'}(x)=gg'x.$$

Thus $g'gx=gg'x$, hence $g'gg'^{-1}g^{-1} \varepsilon G_x$. Therefore G_x^*/G_x is commutative.

Now $\bar{G}_x^* \subset G$ as a closed, therefore Lie, subgroup &, using the compactness of G, $\bar{G}_x^*(x)=\overline{O(x)}$, where $\overline{O(x)}$ is the closure of $O(x)$ in G/H.

Proposition 21

$\bar{G}_x^*/G_x$ is a closed connected Abelian Lie group.

Proof

Let g' & $g \varepsilon \bar{G}_x^*$. Then $\exists$ convergent sequences g_n' & g_n of points of G_x^*, s.t. $g_n' \longrightarrow g'$ & $g_n \longrightarrow g$ respectively. Let $h \varepsilon G_x$.

From Proposition 20, we have $g_n' g_n g_n'^{-1} g_n^{-1}(x)=x$ & so $g_n' g_n g_n'^{-1} g_n^{-1} \longrightarrow g'gg'^{-1}g^{-1} \varepsilon G_x$.

Also $g_n h g_n^{-1} \varepsilon G_x$ so that $ghg^{-1} \varepsilon G_x$.

Thus $G_x \triangleleft \bar{G}_x^*$ & $\bar{G}_x^*/G_x$ is an Abelian Lie group. To see that it is connected, we note that $\overline{O(x)}$ is connected & $\overline{G}^*_x/G_x=\overline{O(x)}$.

Cor 21.1

$\overline{O(x)}$ is an r-torus, T^r, for some r.

Proposition 22

G/H is foliated by r-tori, each of which is invariant by the flow of s.

Proof

We define an equivalence relation * on G/H. Say x*y iff $y\varepsilon\overline{O(x)}$. We assert * is an equivalence relation:

1. Reflexivity. Obvious.

2. Symmetry. If $y\varepsilon\overline{O(x)}$ we wish to show that $x\varepsilon\overline{O(y)}$. Now if $y\varepsilon\overline{O(x)}$ there exists a convergent sequence g_n of elements of G^*_x s.t. $g_n\longrightarrow g$, where $gx=y$. But, therefore, we have a sequence $t_i\varepsilon R$ s.t.:

$$g_i x=F_{t_i}(x)\longrightarrow y \text{ as } i\longrightarrow\infty.$$

Let us consider $F_{-t_i}(y)=g_i^{-1}(y)\longrightarrow g^{-1}y=x$, as $i\longrightarrow\infty$.

i.e. we have shown $x\varepsilon\overline{O(y)}$.

3. Transitivity. Let us suppose x*y & y*z. As above we have sequences t_i, t_i' s.t.:

$$F_{t_i}(x)\longrightarrow y$$

$$F_{t_i'}(y)\longrightarrow z.$$

Given 1/n $\exists j(n)$ s.t. $F_{t'_{j(n)}}(y)$ 1/n approximates z.

And, given $j(n)$, $\exists i(n)$ s.t. $F_{t_{i(n)}+t'_{j(n)}}(x)$ 2/n approximates

$F_{t'_{j(n)}}(y)$. Consider the sequence:

$$F_{t_{i(n)}+t'_{j(n)}}(x) \text{ as } n \longrightarrow \infty.$$ Clearly it has a limit equal to z.

Thus $x*z$.

Consequently we have a partitioning of G/H into disjoint sets of the form $\overline{O(x)}$. Cor. 21.1 shows that we have a partitioning of G/H by tori T^r, where r may not be constant. We will show that r is in fact constant on G/H-even on different connected components of G/H.

Suppose x & y belong to different * equivalence classes. Since G is transitive $\exists g \in G$, s.t. $gx=y$. Consider the map: $E:\bar{G}^*_x \longrightarrow \bar{G}^*_y ; h \longmapsto ghg^{-1}$.

E is clearly a group isomorphism & $E(G_x)=G_{gx}$. Thus E induces an isomorphism $\bar{E}$;

$$\begin{array}{ccc} \bar{G}^*_x/G_x & \longrightarrow & \bar{G}^*_y/G_y \\ \wr\Vert & & \wr\Vert \\ \overline{O(x)} & \longrightarrow & \overline{O(y)} \end{array}$$

giving us the required result.

Thus we have the required foliation of G/H.

Let $s \in C^\infty_G(T(G/H))$ & let $X \in T_x(G/H)$ be G_x-invariant, i.e. $Dg\cdot X=X$ $(g \in G_x)$, then we have:

Proposition 23

The set of G_x-invariant vectors in $T_x(G/H) \cong C^\infty_G(T(G/H))$.

Proof

Given $X \in T_x(G/H)$, with $Dg(X)=X$ $(g \in G_x)$ define s_X by:

$$s_X(fx)=Df(X), f \in G.$$

s_X is clearly well-defined for suppose $fx=hx$, then

$s_X(fx)=s_X(hx)$ iff $Df(X)=Dh(X)$ iff $D(f^{-1}h)X=X$ iff $f^{-1}h\epsilon G_x$. But $fx=hx$ so that this condition is satisfied.

Now $X\longmapsto s_X$ is obviously injective. Conversely any element $s\epsilon C_G^{\infty}(T(G/H))$ defines a unique $X\epsilon T_x(G/H)$ by $X=s(x)$. X is G_x-invariant & this map is an inverse map to the above map.

Now if $x\epsilon G/H, x\epsilon T^r$, where T^r is a torus of the foliation of G/H defined by s.

Let $X\epsilon T_x(T^r)\subset T_x(G/H)$. Now $G_x=id$ on T^r, since G_x is constant on $O(x)$. Thus X defines an equivariant vectorfield on G/H & hence, by restiction, a vectorfield on T^r.

It is not difficult to see that the flow defined by X on T^r is either an irrational flow on $\{y\}\times T^s$, for each $y\epsilon T^{r-s}$, where $T^r=T^{r-s}\times T^s$ or a rational flow on T^r And, in fact, given $s_X\epsilon C_G^{\infty}(T(G/H))$ we may perturb X to X' s.t. $s_{X'}|T^r$ is rational-with arbitrarily high period-or irrational.

Thus we may state:

<u>Theorem 14</u>

If $s\epsilon C_G^{\infty}(T(G/H))$ then:

1. s is C^{∞} & either
2. $s=0$, or
3. G/H is foliated by r-tori, T^r, s.t. $s|T^r$ is isomorphic to a rational or irrational flow, for some r depending on s.

<u>Definition 22</u>

With the above notation, we call r the s-rank of G/H.

We let $R(G/H)=\max\{s\text{-rank}:s\in C_G^{\infty}(T(G/H))\}$. Call $R(G/H)$ the (vectorfield) rank of G/H.

Remarks

1. If $\chi(G/H)\neq 0$, then $R(G/H)=0$.
2. If $\dim(N(H))>\dim H$, then $R(G/H)\geq 1$.

Let $s\in C_G^{\infty}(T(G/H))$ be s.t. $\bar{G}_x^*/G_x\cong T^R$, $R=R(G/H)$. Now $G_x\triangleleft N(G_x)$, define $P=N(G_x)/G_x$. P is a Lie group. We have a natural P-action on the manifold $N(G_x)x$, given by $y\longmapsto[gG_x]y=gy$, $y\in N(G_x)x$.

Now clearly $\bar{G}_x^*\subset N(G_x)$ as a subgroup. Since $G_x\triangleleft\bar{G}_x^*$, we may regard $\bar{G}_x^*/G_x\subset P$ as a closed Lie subgroup.

Theorem 15

$R(G/H)=\operatorname{rank}(N(G_x)/G_x)$, $x\in G/H$.

Proof

$\bar{G}_x^*/G_x$ is a toral subgroup T^R of P. Suppose T^R is not maximal in $N(G_x)/G_x$. Let $T^s\subset P$ be a maximal toral subgroup, then $T^s(x)=T^s$ &, since $T^s(x)\subset Px$, any vector X at x, tangent to $T^s(x)$, defines a vectorfield $\bar{X}$ on $T^s(x)$. By choice of X, $\bar{X}$ may be made into an irrational flow on T^s. As above $\bar{X}$ extends to an equivariant vectorfield on G/H. Thus $R(G/H)\geq s$, contradiction, therefore $s=R$.

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8. Genericity for closed orbits of equivariant vectorfields

Let $s\in C^r_G(TM)$ & let q be a closed orbit of s. We let T denote the prime period of q.

In what follows we will often have to distinguish between the two cases:

A: $G_q/G_x \cong C^r$ & B: $G_q/G_x \cong S^1$.

We have already remarked that if $G_q/G_x \cong S^1$ then $q\subset G(x)$. In particular, q is a C^∞ submanifold of M, since $q=G_q(x)$. If $G_q/G_x \cong C^r$, then q will, in general, be only of class C^r &, as an easy consequence, $G(q)$ will only be of class C^r. We set $\dim(G(x))=p$, where $x\in q$. We note that in case A $\dim G(q)=p+1$.

Let (D,D_o) denote a Poincaré disc pair for q at x (see Abraham 2 for definitions) then we have a Poincare map f, defined by the flow of s, $f:D_o\longrightarrow D$.

Let us consider case B first. We may suppose that $D=R^{p-1}\times R^{n-p}$, where $R^{p-1}\times\{0\}\subset G(x)$ & $\{0\}\times R^{n-p}$ is transverse to $G(x)$ at x.

Locally, we have:

$$Df(x)=\begin{pmatrix} T & X \\ W & Y \end{pmatrix}\in GL(R^{n-1}),\ T\in L(R^{p-1},R^{p-1}),$$

$Y\in L(R^{n-p},R^{n-p})$ etc.

We note that $f:R^{p-1}\times R^{n-p}\longrightarrow R^{p-1}\times R^{n-p}$ is s.t. $f(y,0)=(y,0)$, using the equivariance of the flow of s.

Consequently:

$$Df(x)=\begin{pmatrix} I_{p-1} & X \\ 0 & Y \end{pmatrix}$$

Similarly we have for case A:

$$Df(x)=\begin{pmatrix} I_p & X \\ 0 & Y \end{pmatrix}, \text{where } Y \in L(R^{n-p-1}, R^{n-p-1}) \text{ etc.}$$

Definition 23

We say q is a 2-generic closed orbit of $s \in C^r_G(TM)$ iff, with the above notation, Y has no eigenvalues modulus 1.

Equivalently

1. Noting that $T_x F_t^{(s)} \in L(T_x M, T_x M)$, q is a 2-generic closed orbit iff $T_x F_t^{(s)}$ has $\overline{n-p-1}$ eigenvalues modulus not equal to 1 (Case A) or $\overline{n-p}$ eigenvalues modulus not equal to 1 (Case B).

2. As in the singular point case, it is easy to check that q is 2-generic iff X is G,r-normally hyperbolic on G(q).

We now restrict attention to case A:

Let us consider the G-normal bundle N of G(q), which is the orthogonal complement of TG(q) in $T_{G(q)}M$ w.r.t. an equivariant Riemannian metric. N is of class C^{r-1}.

We have a C^{r-1} G-tubular nbd. K of G(q) in M, constructed using the exponential map, i.e. $K=\exp(N^e)$, where N^e is the set $\{v \in N : \|v\| < e\}$, for some $e>0$. In fact we may regard K as the image of N, by the map exp.C, where C is a suitable equivariant compression. In the sequel we will identify K & N by means of this map.

Now we may define a 'q-slice' at $x \in G(q)$, Q_x, to be equal to N_x. As a consequence of the fact that $N \backslash G(x)$ is a G manifold,

Q_x is a slice in $N|G(x)$ & we have the following properties:

1. $gQ_x \cap Q_x \neq \emptyset$ iff $g \in G_x$.

2. If $p:U \longrightarrow G$ is a local cross section in G/G_x, then the map $F:U \times Q_x \longrightarrow M$, defined by $F(u,s)=p(u)s$ is a diffeomorphism onto an open nbd. of x in N G(x).

3. Q_x is G_x-invariant.

4. We may choose a coordinate system on Q_x s.t.,w.r.t. this system, G_x acts on Q_x as a group of orthogonal transformations.

In the sequel, properties 1 & 3 will be of the most use.

We note that $N|G(x)$ is in fact a C^∞ G vector bundle over G(x), so that q-slices are C^∞.

With the above notation we define diam(K)=diam(N^e)= diam(Q_x)=e.(Not 2e!)

Let us suppose, therefore, that we have the above defined bundle N over G(q). Then, let us set $N^*=N|G(x)$. As indicated above N^* is a C^∞ G vector bundle over G(x).

We observe that, for diam(N) sufficiently small, we have $N^* \pitchfork s$.

Let us define, for $0<r\leq 1$,

$$N_r=\{v \in N^* : \|v\| < re\}.$$

Clearly, for r sufficiently small, we may define, using the flow, a <u>generalised Poincaré map</u>:

$$f : N_r \longrightarrow N^*.$$

Strictly: Define f on some transverse disc nbd. $V \subset N^*$ in usual way, then extend using G: We note that q may meet N^* in more than one point.

<u>Proposition 24</u>

Regarding N_r & N^* as G-manifolds of class C^∞, f is a C^r equivariant diffeomorphism of N_r onto $f(N_r) \subset N^*$.

<u>Proof</u>

Let $y \in N_r$.

$$f(y) = F_{T'(y)}(y), \text{for some } T'(y) \in R, F \text{ denotes the flow of s.}$$

$$= g^{-1} F_{T'(y)}(gy) \text{ - since F is equivariant.}$$

Thus, since N^* is a G-manifold, $F_{T'(y)}(gy) \in N^*$.

Therefore $gf(y) = F_{T'(y)}(gy) = f(gy)$.

We note also that T' is equivariant.

As an immediate consequence of the above we have:

<u>Cor 24.1</u>

If q is a closed orbit of type A, q is 2-generic iff its generalised Poincaré map is 1-generic.

Now if $\dim(N(G_x)) > \dim G_x$, then $\dim(N(G_x)x) > 1$. Since $s \in N^*$, $N(G_x)x \subset N^*$, therefore the following definition makes sense:

<u>Definition 24</u>

If q is a closed orbit of type A, we say q is 2*-generic iff its generalised Poincaré map is 1*-generic.

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9. Some Floquet Theory for Equivariant Vectorfields.

Let $s\epsilon C^r_G(TM)$ & let q be a closed orbit of s & take $x\epsilon q$.

Recall that there are two possibilities:

A: $G_q/G_x \cong C^k, C^k$:cyclic group of order k.

B: $G_q/G_x \cong S^1$.

The aim of this section is to obtain a 'nice' representation of s in a nbd. of q.To do this we adopt a similar approach to that used in Abraham 1,for the G=id case.The fact that,in case A,q may only be of class C^r,& hence Tq of class C^{r-1},necessitates that extra care must be taken in case A to get the strongest results as regards differeniability.Cases A & B will,in fact,often be treated separately.

First we have some preliminaries on bundles.We treat cases A & B separately:

Case A

We define

$$Vq=\bigcup\{T_{gy}(gq):g\epsilon G, y\epsilon q\}.$$

Then Vq=G(Tq),& it is easy to see that Vq may be given the structure of a C^{r-1} G-vector bundle over G(q). (Vq is a sort of vertical tangent bundle of G(q),regarded as 'almost' a fiber bundle over G(x)-it is,if $G_q=G_x$).

We let Nq be the orthogonal complement of Vq in $T_{G(q)}M$, w.r.t. an equivariant Riemannian metric,henceforth supposed fixed.Thus Nq is a C^{r-1} G-vector bundle over G(q).

Let us consider TG(q):This is a C^{r-1} G-vector bundle with a C^{r-1} G-vector bundle complement in $T_{G(q)}M$, say $NG(q)$.

Also we may define:

$$VG=\bigcup\{T_yG(y):y\varepsilon G(q)\}=\bigcup_{x\varepsilon q} TG(x).$$

And again it is easy to see that VG is a C^{r-1} G-vector subbundle of $T_{G(q)}M$.

Now $TG(q)\supset Vq$ as a C^{r-1} G vector subbundle, consequently $NG(q)\subset Nq$ as a C^{r-1} G vector subbundle.

Let us define VG* to be the orthogonal complement of Vq in TG(q). VG* is a C^{r-1} vector bundle. We note that, in general, VG* does <u>not</u> equal VG, regarded as subbundles of $T_{G(q)}M$.

Now if we consider $Nq|q$, we see that Nq is a normal bundle for q &, letting N_q^e denote the open disc bundle radius e, we have that for some $e'>0$, $\exp:N_q^{e'}\longrightarrow M$ is a G_q-invariant diffeomorphism onto a G_q-tubular nbd. of q.

Similarly, NG(q) is a normal bundle for G(q), & $\exists e''>0$, s.t. $\exp:NG(q)^{e''}\longrightarrow M$ is a G-invariant diffeomorphism onto a G-tubular nbd. of G(q).

Thus, taking $0<e\leqslant\min(e',e'')$ & a suitable (equivariant) compression of N_q, we may define both of the above C^{r-1} tubular nbds, in terms of restrictions of the exponential map on $T_{G(q)}M$, since $T_{G(q)}M\supset Nq\supset NG(q)$.

Identifying tubular nbds & corresponding bundles, we note, in particular, that $NG(q)_x$ is a q-slice at x, & that $NG(q)_x$ may be regarded as a G_x-invariant vectorsubspace of Nq_x with complement VG^*_x, also G_x-invariant. Nq_x is a Poincaré disc for q at x, for sufficiently small e.

Remark:

Using Appendix 2, we may take C^r approximations to all the bundles defined above & in particular to Nq & NG(q) & VG* & we may assume $T_{G(q)}M \supset Nq \supset NG(q)$ as a series of C^r G vector bundles & C^r G subbundle inclusions. Everything said above remains true with these C^r approximations, except the tangency properties.

Case B

We may define Vq, Nq & NG(q) as above, noting that all these bundles are now C^∞ & that NG(q)=NG(x), the normal bundle of G(x), since G(q)=G(x). We have the same obsevations regarding normal bundles & tubular nbds, with one new point:

$Vq \subset TG(x)$ as a G-subbundle. We define C to be the orthogonal complement of Vq in TG(x), then we note that $NG(x) \oplus C = Nq$. This is so since $C \perp NG(x)$ & $C \subset Nq$. Now $NG(x)_x$ is a slice at x, not just a q-slice, & further C is tangent to G(x). Thus we may regard the G_x invariant Poincaré disc Nq_x as being the direct sum of the G_x-invariant subspaces C_x & S_x. Pictorially:

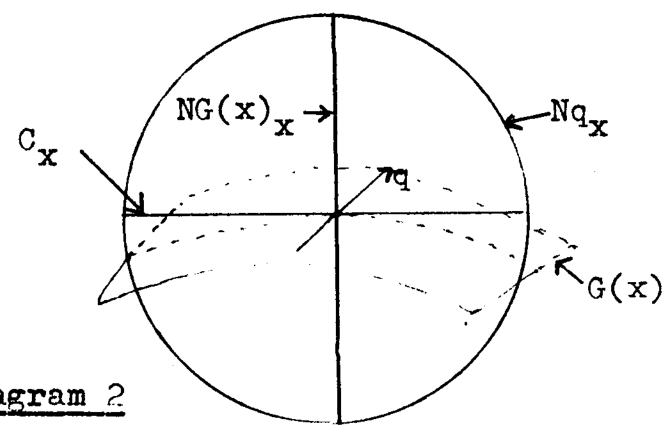

Diagram 2

We will now generalise this tangency property to case A, up to the C^r level. We first prove the following Lemma, a generalisation of a Lemma in P-H-S to the G category.

Lemma 6

Suppose V is a properly embedded G-invariant C^r submanifold of a finite dimensional C^r G manifold M, $r \geqslant 1$. Suppose also that W is another compact C^r G manifold. Let K be a given bundle map, s.t.

$$\begin{tikzcd} T_V M \arrow[r, "K"] \arrow[d] & TW \arrow[d] \\ V \arrow[r, "k"] & W \end{tikzcd}$$ commutes.

We suppose k is a C^r equivariant map & K is a C^{r-1} bundle map : $K|TV=Tk$, & K is equivariant.

Then k extends to $\bar{k}$, where $\bar{k}$ is a C^r eqivariant map of some G-invariant nbd. of V in M into W. Further, $T\bar{k}=K$, on V.

Proof

Let $j:W \longrightarrow R^n$ be a C^r equivariant embedding of W into R^n, where R^n is a Euclidean G-space. Such embeddings exist, see Palais, Borel 1.

As in P-H-S, using the Whitney extension theorem, we may construct a C^r map k^*, $k^*:U \longrightarrow R^n$, s.t. $Tk^*=Tj \cdot K$ on V & U is some nbd. of V in M. Now k^* is not necessarily equivariant but since $Tk^*=Tj.K$ on V, k^* is equivariant on $T_V M$.

Let us take a G-normal bundle p of jW in R^n, this defines a G-tubular nbd. H of jW. W.l.o.g. we may suppose $k^*(U) \subset H$.

We now average k^* over G: define k^+ as follows:

$$k^+(x)=\int_G g^{-1}.k^*(gx)dg.$$

Now $k^*(V)=jW$, so given H we may, for sufficiently small

G-invariant U, insist that $k^*(U)\subset H$. Thus $k^+:U\longrightarrow R^n$ is a C^r equivariant map.

Further, we assert that $Tk^+|T_VM=Tk^*|T_VM$. To see this let $v\in T_VM$ & let $c:R\longrightarrow M$ be s.t. $c'(0)=v$. Then clearly $(gc)'(0)=gv$.

Now $Tk^*(v)=(k^*c)'(0)$. But Tk^* is equivariant on T_VM so that: $Tk^*(v)=g^{-1}Tk^*(gv)=g^{-1}.(k^*gc)'(0)$. Thus:

$$Tk^*(v)=g^{-1}.(k^*gc)'(0).\forall g\in G \quad . \; . \; . \; . \; . \; . \; . \; . \; 1.$$

$$\begin{aligned}
\text{Now } Tk^+(v)=(k^+c)'(0)&=\frac{d}{dt}(k^+c(t))_{t=0}\\
&=\left(\frac{d}{dt}\int_G g^{-1}k^*gc(t)dg\right)_{t=0}\\
&=\int_G g^{-1}.(k^*gc)'(0)dg.\\
&=\int_G Tk^*(v)dg, \text{by } 1.\\
&= Tk^*(v).
\end{aligned}$$

Thus $Tk^+(v)=Tk^*(v)$ $(v\in T_VM)$.

Finally we set $\bar{k}=j^{-1}.p.k^+$. Since p is C^r equivariant & $pk^+(U)=jW$, $\bar{k}$ is well defined & a C^r equivariant map. Since $Tp\ T(jW)=id$, $T\bar{k}=K$ on V.

Now $T_{G(q)}M=Vq\oplus Nq=Vq\oplus NG(q)\oplus VG^*=Vq\oplus NG(q)\oplus VG$, since $Vq\oplus VG^*=TG(q)=Vq\oplus VG$.

Let us take C^r G vector bundle approximations, in $T_{G(q)}M$, $\widehat{NG}(q)$, $\widehat{VG}^*$ & $\widehat{Nq}$ to $NG(q)$, VG^* & Nq respectively, so that $\widehat{NG}(q)$ is still the orthogonal complement of $\widehat{VG}^*$ in $\widehat{Nq}$.

We note that we still have:

$$T_{G(q)}M=Vq\oplus\widehat{Nq}=Vq\oplus\widehat{NG}(q)\oplus\widehat{VG}^*=Vq\oplus\widehat{NG}(q)\oplus VG.$$

We let $N=\widehat{Nq}|q$, $C=\widehat{VG}^*|q$ & $Q=\widehat{NG}(q)|q$; then N, C & Q are C^r

G_q vector bundles over q.

Let us consider the inclusion $i:q\hookrightarrow G(q)$ & the bundle map $K:T_qM \longrightarrow TG(q)$, defined as the composite of K_1 & K_2, where:

1. K_1 is the projection of T_qM onto $Tq\oplus C$, along Q.
2. $K_2:T_qM \longrightarrow T_qM$, is defined by projecting C onto $VG|q$, parallel to $Tq\oplus NG(q)$ q, i.e. $K_2=id$ on $Tq\oplus NG(q)|q$.

Then with the above definition of K, K is a C^{r-1} G_q vector bundle map & $K|Tq=Ti$.

Now, with the notation of Lemma 6, we take $W=G(q)$, $i=k$, & $V=q$ to get a C^r G_q map $\bar{d}:U \longrightarrow G(q)$, where U is a G_q-invariant nbd. of q in M, s.t. $T\bar{d}=K$.

Noting that $NG(q)|q$ is the tangent bundle of $NG(q)|q\subset M$, restricted to q, it follows that we may regard $\bar{C}=\exp(C)|M$ as a locally embedded C^r submanifold of M. (If we recall page 83, we may suppose C is embedded by the exponential map, composed with a suitable compression; we may suppose that this composition 'works' also for Nq, $NG(q)$ etc.). $\bar{C}$ is 'close' to $G(q)$ near q.

We assert that $d'=\bar{d}|\bar{C}:\bar{C} \longrightarrow G(q)$ is a G_q invariant C^r embedding. This is immediate, since $T\bar{d}|T_qC$ is an injection. (We may need smaller U here, of course).

We assert that, given $d':\bar{C} \longrightarrow G(q)$, we may extend d' G_q invariantly to a C^r G_q invariant map of some invariant nbd. V, in M, of q, s.t. $d'|N=id$ (i.e. N embedded in M, see above).

This may be done using Whitney's extension theorem as in Lemma 6 & P-H-S 1:i.e. we embed M equivariantly in R^n_j by the map j,& extend j.d' locally on V,using Whitney's extension theorem as in P-H-S 1 & then we follow the proceedure of Lemma 6.We omit details.

Finally we arrive at a map $d:W \longrightarrow M$,where d is G_q-invariant & C^r & W is a G_q invariant nbd. of q s.t. $d|\bar{C}=d'$ & $d|N=id$.

As on page 83 we may assume C & N both embed inside W.

We now define a new G_q tubular nbd. of q,by composition with d.Thus if $f=\exp.C':N \longrightarrow M$ is the tubular map for N (C' is a compression) we define a new tubular map $h=(d.f):N \longrightarrow M$.This clearly defines a C^r G_q invariant tubular nbd. for q.

We assert that:

1. $h(Q_y)$ is a q-slice at y,$y\varepsilon q$.
2. $h(C_y)$ is tangent to G(y) at $y\varepsilon q$.

1. is clear by construction,since $h|Q=f|Q$.
2. We must show $Td(fC_y)=T_yG(y)$.But $T_y(fC_y)=C_y$ & hence $Td(C_y)=VG_y=T_yG(y)$,by the definition of K & VG.

Thus we have:

Proposition 25_A

Let q be a closed orbit of $s\varepsilon C^r_G(TM)$ of type A.Then there exists a C^r G_q tubular nbd. D of q,s.t.,if N denotes the associated bundle & f the tubular map,then N may be regarded as a codim 1 C^r G_q subbundle of T_qM.Further N is the direct sum of two G_q C^r subbundles Q & C,where,if $x\varepsilon q$,

$T_x(f(C_x))=T_xG(x)$ & $f(Q_x)$ is a q-slice at x.Further Q is the orthogonal complement to C in N.

Remark:

Even though f is C^r it does not follow that s,pulled back to N,is C^r,in fact f_*s will in general be only of class C^{r-1}. In the rest of the section we will often have to assume s is of class at least C^2,if q is an orbit of type A.

The corresponding result for case B is:

Proposition 25_B

Let q be a closed orbit of $s\varepsilon C^r_G(TM)$ of type B.Then there exists a C^∞ G_q tubular nbd. D of q,s.t. if N denotes the associated bundle & f the tubular map,then N may be regarded as a codim 1 C^∞ G_q subbundle of T_qM.Further N is the direct sum of two C^∞ G_q subbundles Q & C where,if $x\varepsilon q$, C_x is a vector subspace of $T_xG(x)$ of codim 1 & $f(Q_x)$ is a slice at x.Also Q is the orthogonal complement to C in N.

Definition 25

We call tubular nbds satisfying the conditions of:
Proposition 25_A:C^r G_q charts for q.
Proposition 25_B:C^∞ G_q charts for q.

We now prove a result corresponding to Proposition 18 for case A.

Proposition 26

If $G_q/G_x \cong C^k$, for some k, then there exists a cyclic subgroup C^m of G_q s.t.:

$$G_q = C^m . G_x \text{ \& } r|m.$$

Proof

We know from the proof of Proposition 16 that $\exists h \in G_q$ s.t. $q \cap G(x) = \{x, hx, .., h^{r-1}x\}$. Now the $h^i, i=0,..,r-1$ clearly belong to different connected components of G_q. If $h^{pr}=id$, for some $p \geq 1$, we set $C^m = \{id, h, .., h^{pr-1}\}$. If not, we may approximate h by h' in G_q s.t. h' belongs to the same component of G_q as h & h' is periodic, i.e. $h'^m = id$ for some $m>0$, for justification of this, see Montgomerry & Zippin 1, page 43 . We assert that h' generates the required group.

We note that $h'x=hx$, since h' belongs to the same component of G_q as h. Thus $h^p = h'^p, p \in Z$. Let C^m denote the cyclic group generated by h'. Clearly $r|m$. We must show that each element k of G_q may be written $k = h'^p g, g \in G_x, 0 \leq p \leq m-1$. This is so since $kx=hx$, for some $p, 0 \leq p \leq m-1$, then $h'^{-p}k \in G_x$. Thus $k = (h'^p)(h'^{-p}k)$.

Definition 26

We say that $P=m/r$ is the period of C^m w.r.t. q.

Now recall that $T_{G(q)}M = NG(q) \oplus TG(q)$ & in case A the bundles on the R.H.S. are C^{r-1} whilst in case B they are C^{∞}, since $G(q)=G(x)$.

Now we may regard NG(q) as being (equivariantly) identified with the quotient bundle:

$$NG(q)=T_{G(q)}M/TG(q)=\bigcup\{T_yM/T_yG(y):y\in G(q)\}.$$

Let F_t denote the flow of s then, since $G(q)$ is F_t invariant both $TG(q)$ & $T_{G(q)}M$ are left invariant by F_t. Thus we may define the quotient map N_zF_t by requiring that the following diagram commutes:

$$\begin{tikzcd} T_zM \arrow[r, "T_zF_t"] \arrow[d, "l"] & T_yM \arrow[d, "l"] \\ NG(q)_z \arrow[r, "N_zF_t"] & NG(q)_y \end{tikzcd} \quad ,y=F_t(z).$$

Here l is the projection map of the quotient.

N_xF_t is of class C^{r-1}, since T_xF_t is of class C^{r-1}.

Let T denote the prime period of q.

Definition 27

We say q is twisted (untwisted) iff $N_xF_T\in L(R^v,R^v)$ is orientation reversing (preserving), here we are identifying R^v with $NG(q)_x$.

Remark: Using the 1-parameter group property of F_t it is not difficult to see that this definition is independent of $x\in q$ (cf. page 75, Abraham 1).

Let us now consider the bundle C, associated with the tubular nbds defined in Propositions 25_A & 25_B. We recall that C was a G_q vector bundle of class C^r in case A & class C^∞ in case B.

Proposition 27

1. In case A: C is a (C^r) trivial bundle over q.
2. In case B: C is a (C^∞) trivial bundle over q.

Proof

We will only prove 1., the proof of 2. is similar. The important part of the proposition is the actual trivialisation constructed.

Now $C=\widehat{VG}^*|q$ & C is C^{r-1} isomorphic to $VG|q$, using a projection map, as used in the proof of Proposition 25_A.

We assert that $\exists$ a C^o-vector bundle isomorphism H', s.t.

$$\begin{tikzcd} VG|q \arrow[d] & S^1 \times R^d \arrow[l, "H'"'] \arrow[d] \\ q & S^1 \arrow[l, "h"'] \end{tikzcd}$$

, commutes

For some fixed $x\in q$, we let $(VG|q)_x$ be identified with R^d, where d is the fiber dimension of VG.

We identify the interval $[0,T)$ with S^1. Here we regard S^1 as the unit circle in the complex plane & identify with $[0,T)$ so as to respect the group actions, i.e. if 'ident' denotes the identification map, ident: $[0,T)\longrightarrow S^1$; $t\longmapsto 2\pi t/T$. Whenever, in the sequel, we identify S^1 with an interval, we shall assume the identification to be of this form.

Let F_t denote the flow of s, then we define:

$$H':S^1 \times (VG|q)_x \longrightarrow VG|q;(t,\dot{x})\longmapsto TF_t(\dot{x}).$$

This is a well defined bundle map, for $G(q)$ & q are F_t-invariant so $VG|q$ is F_t-invariant. Further $F_o=F_T$, so that H' is at least C^o. We note that H' covers h, where $h(s)=F_s(x)$, thus h is C^r.

Therefore $VG|q$, & hence C, is at least C^o trivial & thus C is certainly C^r trivial by a vector bundle morphism C^r

approximation H to H', which we may assume still covers h, since h is C^r. Thus we have the C^r commutative vector bundle diagram:

$$\begin{tikzcd} Q & S^1 \times R^d \arrow[l, "H"'] \arrow[d] \\ q \arrow[from=u] & S^1 \arrow[l, "h"'] \end{tikzcd}$$, where $h(s)=F_s(x)$ &

H is an isomorphism.

In case B, although F_t is only C^r, restricted to q it is C^{∞}, as $F|q$ is isomorphic to the S^1 action on q, $S^1 \subset G_q$.

As an easy corollary to Proposition 27, we see that the definition given of q 'twisted' is equivalent to the usual definition (see Abraham 1, page 75) stated in terms of the normal bundle to q, i.e. $Nq|q$ is orientable iff $NG(q)|q$ is.

Let us consider $NG(q)|q=Q'$, regarded as a C^{r-1} quotient bundle. Fix $x \in q$ & let v=dim. of the fiber of NG(q). We will identify R^v with Q'_x.

We will have to distinguish between the two cases Q' orientable & Q' non-orientable. First let us take an involution J of R^v, which we suppose equal to the identity if Q' is orientable & to be orientation reversing if Q' is non-orientable.

Recall that for orbits of type A or B we have defined a positive integer P, the period of C^m (case A) or S^1 (caseB) w.r.t. q.

Let us take an S^1. We identify S^1 with $[0,PT)$. We define $L:S^1 \times R^v \longrightarrow S^1 \times R^v$ by $L(s,x)=(s+T,Jx)$ we note that $L^P=id$.

Q' orientable

We define a C^o map $A:R\longrightarrow GL^+(R^v)$, where $GL^+(R^v)\subset GL(R^v)$, as those transformations with positive determinant. We require A to satisfy:

1. $A_o=id$.
2. $A_{rT+t}=N_xF_{-rT}\cdot A_t$, $r=1,..,P-1$, $t\in[0,T]$.

Q' non-orientable

We define a C^o map $A:R\longrightarrow GL^-(R^v)$, where $GL^-(R^v)\subset GL(R^v)$, as those transformations with negative determinant. We require A to satisfy.

1* $A_o=J$.

2* $A_{rT+t}=N_xF_{-rT}\cdot A_t\cdot J^r$, $r=1,..,P-1$, $t\in[0,T]$.

That such A exist is a consequence of the path connectedness of $GL^+(R^v)$ & $GL^-(R^v)$.

We define $B_t:Q'_x\longrightarrow Q'_y$, $y=F_t(x)$, $t\in[0,PT]$, by:

$B_t=N_xF_t\cdot A_t$, in the orientable case.

$B_t=N_xF_t\cdot A_t\cdot J$ in the non-orientable case.

Using 2* we may show that, in the non-orientable case, $B_{PT}=J^P$, but $J^P=id$, since P is even for Q' non-orientable as may be easily checked. Define:

$$B':S^1\times R^v\longrightarrow Q';(s,\dot{x})\longmapsto B_s\dot{x}.$$

Since $B_{PT}=id$, B' is continuous, consequently we have a continuous vector bundle map giving us a P-fold cover of Q':

$$\begin{tikzcd} S^1\times R^v \arrow[r, "B'"] \arrow[d] & Q' \arrow[d] \\ S^1 \arrow[r, "b"] & q \end{tikzcd}$$

We note that $b(s)=F_t(x)$ & we also have $B'.L=B'$. To see the latter point, consider $B_{t+T}J\dot{x}=N_xF_{T+t}\cdot A_{t+T}\cdot J(J\dot{x})$, but, using the definition of A_t, we see that the expression on the right hand side $=N_xF_T\cdot N_xF_t\cdot N_xF_{-T}\cdot A_t\cdot J^3\dot{x}=N_xF_t\cdot A_t\cdot J\dot{x}=B_t\dot{x}$, proving that $B'.L=B'$.

Now Q' may be C^r G_q approximated by Q, as in Proposition 25_A (case A), or is a G_q C^{∞} vector bundle Q in case B. Thus we may approximate B' by a C^r vector bundle map B'' (case A) or a C^{∞} vector bundle map B'' (case B) covering b. b. Now in general $B''.L\neq B''$, but define:

$$B(s,\dot{x})=\Big(\sum_{r=0}^{P-1} B''(s+rT,J^r\dot{x})\Big)/P.$$

Then $B.L=B$ & B is s.t.:

$$\begin{tikzcd} S^1\times R^v \arrow[r, "B"] \arrow[d] & Q \arrow[d] \\ S^1 \arrow[r, "b"] & q \end{tikzcd}$$

is a P-fold cover of Q, C^r in case A, C^{∞} in case B.

Now B is a vector bundle immersion &, using B, we will pull back the G_q-action on Q to one on $S^1\times R^v$.

<u>Case A</u>

We define a G_x-action on $S^1\times R^v$ by:

$g(s,t)=B_s^{-1}(g(B(s,t)))$, where $B_s=B|\{s\}\times R^v$, $g\in G_x$ & $(s,t)\in S^1\times R^v$.

Clearly this gives us an induced C^r G_x-action on $S^1\times R^v$. To define the C^m action on $S^1\times R^v$, we first note that if h generates C^m & $x\in q$, then $hx=F_{T/r}(x)$. We define h on $S^1\times R^v$ by:

$$h(s,t)=B^{-1}_{s+T/r}(hB(s,t)).$$

Then $h^p(s,t)=B^{-1}_{s+pT/r}(h^pB(s,t)),p\varepsilon\{0,..,p-1\}$,& we note that $h^m=id$ & the C^m-action on $S^1 \times R^v$ is free.

Regarding, for the moment, $C^m \times G_x$ as a set, we define a $C^m \times G_x$ action on $S^1 \times R^v$ by setting:

$$(h^p,g)(s,t)=h^p(g(s,t)),h^p\varepsilon C^m,g\varepsilon G_x \quad . \; . \; . \; . \; . \; 1.$$

Now $C^m \times G_x$ does not give a Lie group action on $S^1 \times R^v$, if regarded as a product, howevere, using the 'semi-direct product construction', see Scott 1, page 101, we take the group composition on $C^m \times G_x$ defined by:

$$(u,v)(u',v')\longmapsto(uv,u'^{-1}vu'v'),u,u'\varepsilon C^m;v,v'\varepsilon G_x.$$

Then $C^m \times G_x$, with this action, is a Lie group & acts as a Lie group of C^r transformations on $R \times R^v$, using the action defined in 1., since:

$$(h^p,g)((h^q,g')(s,t))=h^pgh^qg'(s,t)=(h^{p+q},h^{-q}gh^qg')(s,t)$$
$$=((h^p,g)(h^q,g'))(s,t).$$

We will denote $C^m \times G_x$, with this composition, by G^*_q.

We assert that, w.r.t. G^*_q & G_q, B is an equivariant map, in the following sense: If $(h^p,g)\varepsilon G^*_q$, then:

$$B((h^p,g)(s,t))=h^pgB(s,t).$$

This is so for:

$$\begin{aligned}B((h^p,g)(s,t))&=B(h^p(g(s,t)))\\&=B(h^p(B^{-1}_s(gB(s,t))))\\&=B(B^{-1}_{s+Tp/r}(h^pB(B^{-1}_s(gB(s,t)))))\\&=h^pgB(s,t)\end{aligned}$$

Case B

We proceed as for case A, first defining the G_x-action on $S^1 \times R^v$ then the S^1 action. We note that the action is C^∞ here as B is C^∞. Also S^1 acts freely on $S^1 \times R^v$. We set $G_q^* = S^1 \times G_x$, with the above composition. Again B is an equivariant map in the sense of case A.

Remark:

1. In either case A or case B, $S^1 \times R^v$ is a G_q^* vector bundle over R, since B is a vector bundle map.

2. We note that 'L' commutes with the G_q^*-action on $S^1 \times R^v$; this follows since L obviously commutes with the C^m & G_x actions on $R \times R^v$, since B is L-invariant, thus L commutes with the G_q^* action.

Now recall that in:

<u>Case A</u> $N=Q\oplus C$, where N, Q & C are the C^r bundles defined in Proposition 25_A.

<u>Case B</u> $N=Q\oplus C$, where N, Q & C are the C^∞ bundles defined in Proposition 25_B.

We showed, in Proposition 27, that C was C^r (C^∞) trivial in case A (case B). We may clearly take a P-fold cover of C & pull back the G_q action, as done above, to get:

$$\begin{tikzcd} C \arrow[d] & S^1 \times R^d \arrow[l, "A"'] \arrow[d] \\ q & S^1 \arrow[l, "b"'] \end{tikzcd} \quad \text{where } b(s)=F_s(x).$$

We have a G_q^* action on $S^1 \times R^d$ & C^m (or S^1) acts freely on $S^1 \times R^d$. This action is C^r in case A, C^∞ in case B, & in either case $S^1 \times R^d$ is a G_q^* vector bundle.

Thus we have a P-fold cover $H=B\oplus A$ of $Q\oplus C$, s.t.

$$\begin{tikzcd} N=Q\oplus C \arrow[d] & S^1 \times R^v \times R^d \arrow[l, "H"'] \arrow[d] \\ q & S^1 \arrow[l, "h"'] \end{tikzcd} \quad , h(s)=F_s(x),$$

C^m (S^1) acts freely on $S^1 \times R^v \times R^d$ & $S^1 \times R^v \times \{0\}$ & $S^1 \times \{0\} \times R^d$ are mapped onto Q & C respectively by H. Further $S^1 \times R^v \times \{0\}$ & $S^1 \times \{0\} \times R^d$ are G_q^* invariant subbundles of the G_q^* vector bundle $S^1 \times R^v \times R^d$.

We define J on $R^v \times R^d$ by $J(x,y)=(Jx,y)$, then J is an involution. If we define $L: S^1 \times R^v \times R^d \longrightarrow S^1 \times R^v \times R^d$ by $L(s,x,y)=(L(s,x),y)$, then $H.L=H$, & also L commutes with G_q^*.

We now take the universal cover of $S^1 \times R^v \times R^d$ to get the commutative vector bundle diagram:

$$\begin{tikzcd} N=\Phi C \arrow[d] & S^1 \times R^v \times R^d \arrow[l, "H"'] \arrow[d, "\pi_1"] & R \times R^v \times R^d \arrow[l, "(c,id)"'] \arrow[d] \\ q & S^1 \arrow[l, "h"'] & R \arrow[l, "c"'] \end{tikzcd},$$

where $c(t)=t \bmod PT$, & c is C^{∞}.

We first pull back the G_q^* action on $S^1 \times R^v \times R^n$ to an action on $R \times R^v \times R^d$:

Case A

First we define a Z-action on $R \times R^v \times R^d$:

If $n \in Z$, we define $n(s,(x,y))$, $(s,(x,y)) \in R \times R^v \times R^d$, by

$$n(s,(x,y))=(c,id)^{-1}_{s+nT/r}(h^n(c,id)(s,(x,y)),\text{ where}$$

$(c,id)_{s+nT/r}=(c,id)|\{s+nT/r\} \times R^v \times R^d$ & h is the generator of C^m.

We similarly define a G_x-action on $R \times R^v \times R^d$, by pulling back the G_x-action on $S^1 \times R^v \times R^d$, using (c,id).

We define a composition on $Z \times G_x$, making it into a Lie group, by:

$$(n,g)(n',g')=(n+n',h^{-n'}gh^{n'}g'),\ n,n' \in Z;\ g,g' \in G_x.$$

We denote $Z \times G_x$, with this composition, by $G_q^!$. Then $G_q^!$ acts as a Lie group of C^r transformations on $R \times R^v \times R^d$ by:

$$(n,g)(s,(x,y))=n(g(s,(x,y))).$$

Then (c,id) is an equivariant vector bundle map, w.r.t. $G_q^!$ & G_q^*, i.e: $(c,id)(n,g)(s,(x,y))=(h^n,g)(c,id)(s,(x,y))$.

Case B

Here we have a C^{∞} $R \times G_x = G_q^!$ action on $R \times R^v \times R^d$, defined as above.

In both cases A & B:

1. Set $K=H.(c,id)$. Then K is an equivariant VB-immersion, i.e. if $(n,g) \in G_q^!$, then $K(n,g)=h^n gK$.

K is of class C^r (case A) & C^∞ in case B.

2. $R \times R^v \times R^d$ is a $G_q^!$ vector bundle with $G_q^!$ invariant complememtary (orthogonal) subbundles $R \times R^v \times \{0\}$,& $R \times \{0\} \times R^d$.$G_q^!$ acts as a C^r group of transformations in case A & as a C^∞ group in case B.Further Z (R in case B) acts freely on $R \times R^v \times R^d$.

3. K.L=K & also L commutes with the $G_q^!$ action.

4. $K(R \times R^v \times \{0\})=Q$ & $K(R \times \{0\} \times R^d)=C$.

5. $k(s)=F_s(x)$,where K covers $k:R \longrightarrow q$.

Definition 28

We say that the triple $(R \times R^v \times R^d, G_q^!, K)$ is a C^r (case A),C^∞ (case B) 'G_q pseudo chart for q',a 'GPC',if $R \times R^v \times R^d$ is a universal cover for N,satisfying conditions 1,2,4,5 above.If,in addition,it satisfies condition 3 we call it a demi-periodic GPC.

Remark:As we have defined it,$G_q^!$ is not unique,as m (or P) is not unique.We could have insisted that m was a minimum & pursued uniqueness properties.We will not do this here.

Now given the tubular nbd. D of q & associated bundle $N=Q\oplus C$ & tubular map f,we may pull back s on D to a vector field,which we still denote by s,on N:s,pulled back is only of class C^{r-1},if q is of type A.

Now we have the vector bundle immersion K &

$TK_{(s,x,y)}:R \times R^v \times R^d \longrightarrow T_{K(s,x,y)}N$ is a vector space isomorphism,so we may pull back s to K_*s,where:

$$(K_*s)(s,x,y)=(TK_{(s,x,y)})^{-1}s(K(s,x,y)).$$

We will denote K_*s by s^*.

Theorem 16

Let M be a G-manifold, $s \in C_G^r(TM)$ & q a closed orbit, prime period T (If q is of type A, we suppose $r \geq 2$). Then there exists a demi-periodic GPC $(R \times R^v \times R^d, G_q^!, K)$ for q s.t. the principal parts of the local representitive of s, w.r.t. the GPC, $s^* = (s_1^*, s_2^*, s_3^*)$, where:

$$s_1^*: R \times R^v \times R^d \longrightarrow R$$
$$s_2^*: R \times R^v \times R^d \longrightarrow R^v$$
$$s_3^*: R \times R^v \times R^d \longrightarrow R^d ,$$

have the form:

$$s_1^*(t,x,y) = 1 + Q(t,x,y)$$
$$s_2^*(t,x,y) = A(t)x + R_1(t,x,y)$$
$$s_3^*(t,x,y) = B(t)x + C(t)y + R_2(t,x,y),$$

where $(t,x,y) \in R \times R^v \times R^d$ &:

1. $A(t) \in L(R^v, R^v)$.
2. $B(t) \in L(R^v, R^d)$.
3. $C(t) \in L(R^d, R^d)$.
4. $Q(t,0,0) = 0, \forall t \in R$.
5. $R_1(t,0,0) = 0; R_2(t,0,0) = 0, \forall t \in R$.
6. $D_i R_j(t,0,0) = 0, i = 2$ or $3, j = 1$ or 2.

Also we have the following relations involving J & T:

1* $A(t+T) = JA(t)J$.

2* $B(t+T) = B(t)J$.

3* $C(t+T) = C(t)$

4* $Q(t+T,x,y) = Q(t,Jx,y)$.

5* $R_1(t+T,x,y) = JR_1(t,Jx,y)$.

6* $R_2(t+T,x,y) = R_2(t,Jx,y)$.

Proof

We take as GPC the demi-periodic GPC constructed above. First we note that $s^*(t,0,0) = (1,0,0)$. This follows as in

Abraham 1, as our demi-periodic GPC is certainly a pseudo-chart in the sense of Abraham, thus:

$$s_1^*(t,x,y)=1+Q(t,x,y),\text{with } Q(t,0,0)=0.$$

Now, if q is of type B, s* is of class C^r. If q is of type A, s* is of class C^{r-1}. Thus, in the latter case we assume $r\geq 2$. With this proviso, we apply Taylor's theorem to get:

$$s^*(t,x,y)=\begin{pmatrix}1+Q(t,x,y)\\ 0\\ 0\end{pmatrix}+\begin{pmatrix}0 & 0 & 0\\ 0 & Ds_2^*(t,0,0) & \\ 0 & Ds_3^*(t,0,0) & \end{pmatrix}\begin{pmatrix}0\\ x\\ y\end{pmatrix}+\begin{pmatrix}0\\ R_1(t,x,y)\\ R_2(t,x,y)\end{pmatrix}$$

R_1 & R_2 satisfy conditions 5 & 6.

We consider $Ds_2^*(t,0,0)$. First:

$$Ds_2^*(t,0,0)\begin{pmatrix}x\\ y\end{pmatrix}=A(t)x+M(t)y,\ A(t)\in L(R^v,R^v),\ M(t)\in L(R^d,R^v).$$

Now $C_y\subset T_yG(y)$, $y\in q$ & thus, to linear approximation, we may regard $C_y\subset G(y)$. Consequently s resticts to a vector field on $C\subset Q\oplus C$ - at least to linear approximation. But, since K respects the splitting $Q\oplus C\xrightarrow{K}R\times R^v\times R^d$, s* restricts to a vector field on $R\times\{0\}\times R^d$, i.e.:

$$s^*:R\times\{0\}\times R^d\longrightarrow R\times\{0\}\times R^d\subset R\times R^v\times R^d.$$

(again to linear approximation), thus $M(t)=0$. Thus conditions 1,..,6 are true.

Since $K.L=K$, $L_*K_*=K_*s$, thus $L_*s^*=s^*$ & this implies:

$$s_1^*(t+T,x,y)=s_1^*(t,Jx,y),$$
$$s_2^*(t+T,x,y)=Js_2^*(t,Jx,y),$$
$$s_3^*(t+T,x,y)=s_3^*(t,Jx,y).$$

Thus we have the additional relations 1*,..,6*.

Next we aim to remove the dependence of the linear approximation $A(t)$ to s_2^* on t. We will do this by defining a map $T:R \times R^v \times R^d \longrightarrow R \times R^v \times R^d; (t,x,y) \longmapsto (t,P(t)x,y)$, where $P:R \longrightarrow L(R^v,R^v)$ is s.t.:

1. $P(t)$ is an isomorphism for each $t \in R$.
2. $P(t)$ is periodic, period $2T$.
3. $P(t).A(t).P(t)^{-1}+P'(t).P(t)^{-1}$ is independent of t.

This is a simple variant of Floquets theorem, the proof given here is similar to that in Abraham 1:

We define a vectorfield w on $R \times R^v \times R^d$ by setting

$$w(t,x,y)=(t,x,y;1,A(t)x,0)$$

It is clear that:

1. w is complete.
2. The flow of w is of the form $H(s,x,y;t)=(s+t,G_t(s),y)$, where $G_t(s) \in L(R^v,R^v)$ is an isomorphism & the map $t \longrightarrow G_t(s)$ is a 1-parameter group, i.e.:

$$G_{t+r}(s)=G_t(s+r).G_r(s).$$

As $A(t)$ is demi-periodic, so is w, i.e. $L_*w=w$. Consequently L commutes with the flow of w or:

$$G_t(s+T)=JG_t(s)J, t \in R, s \in R.$$

Using the group property of G_t we see that:

$$G_{2T}(0)=JG_T(0)JG_T(0).$$

Therefore $G_{2T}(0)$ has a square root, but an isomorphism $Y \in L(R^v,R^v)$ has a square root iff it has a logarithm (Pontragin 1, page 285). Thus there exists an $A \in L(R^v,R^v)$ s.t.

$$\exp(2TA)=G_{2T}(0).$$

Define $P(t)=\exp(tA).(G_t(0))^{-1}$. P clearly satisfies conditions 1 & 2.

We consider $T:R \times R^v \times R^d \longrightarrow R \times R^v \times R^d;(t,x,y) \longmapsto (t,P(t)x,y)$.

Then $T^*w(t,x,y)=DTw(T^{-1}(t,x,y)$, now:

$$DT_{(t,x,y)}=\begin{bmatrix} 1 & 0 & 0 \\ P'(t)(\cdot)x & P(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, therefore, $T^*w(t,x,y)=$

$$\begin{bmatrix} 1 & 0 & 0 \\ P'(t)(\cdot)P(t)^{-1}x & P(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}\left\langle \begin{matrix} 1 \\ A(t)P(t)^{-1}x \\ 0 \end{matrix} \right\rangle =$$

$$\left\langle \begin{matrix} 1 \\ P'(t)P(t)^{-1}x+P(t)A(t)P(t)^{-1}x \\ 0 \end{matrix} \right\rangle$$

But the integral curve of w, through $(0,x,y)$ is mapped by T into the curve $t \longmapsto (t,\exp(tA)x,y)$. Consequently:

$$P(t).A(t).P(t)^{-1}+P'(t).P(t)^{-1}=A, \text{proving } 3.$$

Theorem 17

There exists a GPC $(R \times R^v \times R^d, G_q^!, S)$ for q, s.t. the principal parts of the local representitive of s^*, w.r.t. this pseudo-chart, $s^*=(s_1^*,s_2^*,s_3^*)$ have the form:

$$s_1^*(t,x,y)=1+Q(t,x,y)$$

$$s_2^*(t,x,y)=Ax+R_1(t,x,y)$$

$$s_3^*(t,x,y)=B(t)x+C(t)y+R_2(t,x,y), \text{where:}$$

$A\in L(R^v,R^v)$; $Q(t,0,0)=0$, $R_i(t,0,0)=0$, $i=1$ or 2,

$D_jR_i(t,0,0)=0, i=1$ or $2, j=2$ or 3.

We also have that Q, R_1, R_2, B & C are periodic in t, period $2T$.

Proof

Using the map T defined above, $T: R \times R^v \times R^d \longrightarrow R \times R^v \times R^d$ we push forward the $G_q^!$ action using T, so as to make T an equivariant vector bundle map & $R \times R^v \times R^d$ a $G_q^!$ vector bundle over R.

We note that T respects the splitting $R \times R^v \times \{0\}$, $R \times \{0\} \times R^d$. Thus $(R \times R^v \times R^d, G_q^!, K.T)$ is a GPC for q.

Working with the linear approximation to s^*, we see that:

$$T^*s^*(t,x,y)=\begin{bmatrix} 1 & 0 & 0 \\ P'(t)(\cdot)P(t)^{-1}x & P(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ A(t)P(t)^{-1}x \\ B(t)P(t)^{-1}x+C(t)y \end{pmatrix}$$

$$=\begin{pmatrix} 1 & \\ 0 + & Ax \\ 0 & \bar{B}(t)x+C(t)y \end{pmatrix}.$$

We note that we get a contribution to s_2^* from s_1^*, namely $(P'(t)Q(t,P(t)^{-1}x,y))P(t)^{-1}x$. However, since Q is C^1 in x & y, this does not contribute to the linear approximation & satisfies the conditions for R_1. Thus the theorem is proved.

Remark:

Before proceeding, it might be worth asking how far can theorem 16 be strengthened. First it would seem possible to insist in Theorem 15 that $C(t)\equiv 0$, i.e. in the trivialisation $S^1 \times R^d$ of C, if $S^1 \times \{s\}$, $s \in R^d$, corresponds, at least to linear approximation, to $gq, g \in G$, then we would have $C(t)\equiv 0$ (This is easy to do, though with a loss of differentiability for type A closed orbits). One might then try & remove the dependence on 't' of both A & B. Again, this is not difficult to do, but the map T no longer preserves the splitting $Q \oplus C$, though it still preserves C. As we need this splitting for our main application of Theorem 16, we will not pursue this point here, but refer the reader to Appendix 4.

Definition 29

Let q be a closed orbit of $s \in C^r_G(TM)$, if A has no eigenvalues real part zero in the representation of Theorem 16, we say q is 2"-generic.

It is not a priori obvious that this definition is well defined, i.e. independent of the particular Floquet representation chosen. The next Lemma resolves this point & shows that 2"-genericity is equivalent to 2-genericity. The Lemma is adapted from a Lemma in Abraham 1

Lemma 7

Let s* be the vector field defined on the Floquet pseudo-chart of Theorem 16. Let H denote the flow of s*,

$$H_t(r,x,y) = (A^1_t(r,x,y), A^2_t(r,x,y), A^3_t(r,x,y)), \text{ where}$$

$(r,x,y) \in R \times R^v \times R^d$.

Then $D_2A^2_t(r,0,0) = \exp(tA)$.

Proof

Regard $H_t(r,x,y)$ as a function of t & x, i.e. we suppose r & y fixed.

Since $A_0^2 = id, D_2A_0^2(r,0,0) = id$. It clearly suffices to show:

$$\frac{d}{ds}\left\{D_2A_s^2(r,0,0)\right\}_{s=t} = D_2A_t^2(r,0,0).A.$$

Now it is easy to see, interchanging the order of differentiation, that:

$$\frac{d}{ds}\left\{D_2A_s^2(r,0,0)\right\}_{s=0} = A.$$

Using the 1-parameter group property & the chain rule:

$$D_2A_{t+s}^2(r,0,0) = D_2A_t^2(A_s^2(r,0,0)).D_2A_t^2(r,0,0)$$

Hence:

$$\begin{aligned}\frac{d}{ds}\left\{D_2A_s^2(r,0,0)\right\}_{s=t} &= \frac{d}{ds}\left\{D_2A_{t+s}^2(r,0,0)\right\}_{s=0}\\ &= \frac{d}{ds}\left\{D_2A_t^2(A_s^2(r,0,0)).D_2A_s^2(r,0,0)\right\}_{s=0}\\ &= \frac{d}{ds}\left\{D_2A_t^2(A_s^2(r,0,0))\right\}_{s=0}.D_2A_0^2(r,0,0)+\\ &\quad D_2A_t^2(r,0,0).\frac{d}{ds}\left\{D_2A_s^2(r,0,0)\right\}_{s=0}.\end{aligned}$$

But the first term on the R.H.S. clearly $=0$, as $A_s^2(r,0,0)=0$, therefore:

$$\begin{aligned}\frac{d}{ds}\left\{D_2A_s^2(r,0,0)\right\}_{s=t} &= D_2A_t^2(r,0,0).\frac{d}{ds}\left\{D_2A_s^2(r,0,0)\right\}_{s=0}\\ &= D_2A_t^2(r,0,0).A\end{aligned}$$

Now the flow H_t defines a local C^r diffeomorphism $F:U\subset\{0\} \times R^v \times R^d \longrightarrow \{2T\} \times R^v \times R^d$, where U is a nbd. of $(0,0,0)$. Noting that $H_{2T}(0,0,0)=(2T,0,0)$, Lemma 7 shows us that $D_2F_2(0,0)=\exp(2TA)$.

Recalling that $f(C)$ is tangent to $G(q)$ at q & that $T_qG(q)$ is invariant by the differential of the flow, we see that:

$$DF(0,0)=\begin{pmatrix}\exp(2TA) & 0\\ L & M\end{pmatrix} \quad \ldots\ldots\ldots\ldots 1$$

Now the map F is the pull back of j^2, where j is the square of the Poincare map for q, defined on the Poincare disc $N_x=Q_x \oplus C_x$, where $S(0,0,0)=K.T(0,0,0)=x$, i.e.:

$$S_{2T}.F=j^2.S_0, \text{where } S_s=S|\{s\} \times R^v \times R^d.$$

Thus $S_{2T}.DF(0,0)=Dj(x).S_0$.

Since S is periodic, period $2T$, we have immediately:

<u>Lemma 8</u>

q is 2-generic iff q is 2"-generic.

Our objective is now to show that given a closed orbit q of $s\varepsilon C^r_G(TM)$, of type B, we have arbitrarily small C^r equivariant perurbations s' of s, s.t. q becomes a 2-generic closed orbit for s'. For type A closed orbits, using this approach, we only get C^{r-1} perturbations for $s, r\geqslant 2$.

Suppose X is a vectorfield defined on the demi-periodic GPC $(R \times R^v \times R^d, G_q^!, K)$. We ask when $X=K_*Y$, for some vectorfield Y defined on N. We assert that it is necessary & sufficient that $L_*X=X$; For $L_*K_*=K_*$, so if X is the lift of Y, $L_*X=X$, so the condition is necessary. Conversely, given X, define Y on N by $Y(z)=TK.X(z')$, where $z'\varepsilon K^{-1}(z)$. Now if $z'\varepsilon K^{-1}(z)$, so does Lz', for $K.L(z')=K(z')$. Therefore Y is well defined & the condition is sufficient.

Now if X is C^s, Y will be of class C^s, in case B, since K is of class C^∞ (In case A Y will only be of class $C^{\min(r-1,s)}$).

We restrict attention to case B:

Let us define $X \in C^\infty(T(R \times R^v \times R^d)$ by:

$$X(t,x,y)=(t,x,y;0,F(t,x)x,0), \text{where}$$

$F: R \times R^v \longrightarrow L(R^v,R^v)$ is a C^∞ map. Let us suppose that $JF(t+T,Jx)Jx=F(t,x)$, then $L_*X=X$ & X is the lift of a C^∞ vector field Y, defined on N:

$$Y(z)=TK.(0,F(t,x)x,0), \text{where } (t,x,y)\in K^{-1}(z).$$

Now, using the Floquet map T, we consider T^*X. From the formula on page 104. it is clear that:

$$T^*X(t,x,y)=(t,x,y;0,P(t).F(t,P(t)^{-1}x)P(t)^{-1}x,0).$$

We now define a map $f: R \times R^v \times \{0\} \longrightarrow R$, s.t.:

1. f is C^∞.
2. f is $G_q^!$-invariant, i.e. $f(g(t,x))=f(t,x)$; $g\in G_q^!$; $(t,x)\in R \times R^v$.
3. f is L-invariant, i.e. $f(t+T,Jx)=f(t,x)$.
4. $f=1$, on some nbd. E_1 of $R \times \{0\}$ in $R \times R^v$.
5. $f=0$, outside of some nbd. E_2 of $R \times \{0\}$ in $R \times R^v$.

That such a map exists may be seen by defining a map f' satisfying 1, 4 & 5 on the P-fold cover $S^1 \times R^v$ of C. We then average w.r.t. the group $Z_P \times G_q^*$ (where Z_P is the action corresponding to L, -recall L commutes with G_q^*). Finally we pull back to a map on $R \times R^v$ satisfying 1,..,5.

We set $F: R \times R^v \longrightarrow L(R^v,R^v); (t,x) \longmapsto (f(t,x)I)x$.

Then F satisfies the above conditions & defines a C^r G_q'-invariant vector field $Z=T^*X$ on the Floquet chart $R \times R^v \times R^d$ s.t.:

$$Z(t,x,y)=(t,x,y;0,f(t,P(t)^{-1}x)Ix,0).$$

Thus on some nbd. F_1 of the zero section in $R \times R^v \times R^d$ we have $Z(t,x,y)=(t,x,y;0,Ix,0)$ & Z is supported on some bounded nbd. F_2 of the zero section. We note that $\|Z\|_r<\infty$.

Define $Z_k=kZ.k\varepsilon R^+$. We define:

$s_k^*=s^*-Z_k$, then we have:

$$D(s_k^*)_2(t,0,0)=(A-kI).$$

Now if q is 2-generic there is nothing to prove, so suppose q is not 2-generic. Then $\exists h>0$, s.t. if $0<k<h$, $A-kI$ is generic, i.e. has no eigenvalues real part zero.

Now let us consider s_k, defined on the tubular nbd. D of q. We note that $s_k=s$ on a collar of the boundary of D in M.

We recall that $f(Q_y)$, $y\varepsilon q$, was a slice at y, which we will denote by Q_y'. Set $W=\bigcup_{y\varepsilon q} Q_y'$.

Let us consider $s_k|W$. We note that W is G_q-invariant, we extend $s_k|W$ to a tubular nbd. of $G(y)$, $G(W)$, by:

$$\bar{s}_k(gz)=gs_k(z), g\varepsilon G, z\varepsilon W.$$

Now since $s_k|W$ is G_q-invariant, it is easy to check that $\bar{s}_k$ is well defined & G-invariant. If s_k is C^r so is $\bar{s}_k$. Since $s_k=s$ on a collar of the boundary of W, $\bar{s}_k$ extends to $s_k'\varepsilon C_G^r(TM)$, with $s_k'=s$ on $M-G(W)$.

We assert that q is a 2-generic closed orbit for s_k',

$0<k\leq h$. We just work backwards: Compute s_k' in the Floquet chart constructed for s. Denote the local representitive of s_k' in this chart by $(s_k')^*$. The fact that $f(Q_y)$, $y\in q$, is a slice at y, means that $(s_k')^*|R \times R^v \times \{0\}$ is unchanged, i.e.:

$$D(s_k')_2^*(t,0,0)=A-kI.$$

Thus q is a 2"-generic closed orbit for s_k' & hence, by Lemma 8, q is a 2-generic closed orbit for s_k'.

To see that we may insist on s_k' being arbitrarily C^r close to s, just note that $\|Z\|_r<\infty$, implies that, as $k\longrightarrow 0$, $\|kZ\|_r\longrightarrow 0$, & hence $\|s_k'-s\|_r\longrightarrow 0$.

Thus we have proved, for type B closed orbits:

<u>Theorem 18</u>

Let $s\in C_G^r(TM)$ & let q be a closed orbit of s. Then we may arbitrarily C^r approximate s by s', s.t. q is a 2-generic closed orbit for s'.

Now the above approach, for type A closed orbits, will only give a C^{r-1} approximation to s, which is 2-generic $(r>2)$. This occurs, since the tubular map is only of class C^r as is N. Thus we cannot even <u>define</u> C^r approximations to s on N.

However Theorem 18 is easily proved for type A closed orbits, without loss of differentiability, & we give a proof in Appendix 3.

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10. Some Perturbation theory for neighborhoods of closed orbits of Equivariant vectorfields.

This section is divided into two halves, the first half is concerned with closed orbits of type B, the second with orbits of type A. In both instances we will be proving a result about genericity in a nbd. of the closed orbit rather than just for the orbit as in section 9. We will eventually use the results of this section to prove a density theorem for 2-generic vectorfields & diffeomorphisms.

Case B

Let q be a closed orbit of $s \varepsilon C^r_G(TM)$ of type B. Let $x \varepsilon q$.

Definition 30

If q is 2-generic & $R(G/G_x)=1$, we say q is 2*-generic.

Suppose $w \varepsilon C^\infty_G(G/G_x)$ & c is a closed orbit of w. Then we will show that if $R(G/G_x)>1$, we may perturb w to w' s.t. all w' orbits are non-compact.

To prove this it would clearly be sufficient to find some vectorfield $k \varepsilon C^\infty_G(T(G/G_x))$ s.t. k gives rise to a foliation of G/G_x by r-tori, $r>1$, & s.t. c is contained in one of these r-tori; For then we could apply theorem 14.

But for each maximal torus $T^r \subset N(G_x)/G_x$ we have a foliation of G/G_x defined by $T^r(x)$. Now we may certainly choose a maximal torus T^r in $N(G_x)/G_x$ s.t. $T_x(T^r(x)) \supset T_x c$, & this torus will suffice.

Now suppose q is a (2-generic) closed orbit of type B, for $s \varepsilon C^r_G(TM)$.

Proposition 28

With the above notation, if $R(G/G_x)>1$, there exists arbitrarily small C^r perturbations s' of s, s.t. $G(x)$ is invariant by s' & $G(x)$ is filled with non-compact orbits.

Proof

The above shows that, restricted to $G(x)$, we can certainly so perturb s. The result is then immediate using Lemma 2.

Let $x\in M$ be of type i, thus $x\in M_i$. Let S be a slice at x in M_i. Thus if $y\in S, G_y=G_x$.

For $y\in S$, let us define $N_y\subset G(y)\subset M_i$, by:

$$N_y=\{z\in G(y):G_z=G_x\}, \text{ thus } N_y=N(G_x)y.$$

Define $H=\bigcup_{y\in S} N_y$, then H may be given the structure of a C^∞ fiber bundle over S:

$$\begin{tikzcd} S\times N_x \arrow[r, "k"] \arrow[d, "\pi_1"'] & H \arrow[dl, "p"] \\ S & \end{tikzcd} \qquad \begin{array}{l} k(s,gx)=gs. \\ p(N_y)=y. \end{array}$$

Since H is a fiber bundle over S we may define its vertical tangent bundle, VTH. We have that VTH is a subbundle, & consequently a submanifold of TH.

Now suppose $\dim H_x=p$, $\dim S=q$, then $\dim H=p+q$, $\dim VTH=2p+q$ & $\dim TH=2(p+q)$.

We choose a further fixed slice $S'\subset S$ of, say, half the diameter of S. Suppose $X:S\longrightarrow T_SM$ is a C^r vectorfield s.t.:

1. X is supported on some $\bar{S}''$, where $\bar{S}''\subset S$ & S'' is a slice at x.

2. X is G_x invariant.

We note that X is G_x-invarinant iff X takes values in

$T_S H$.

Lemma 2 allows us to extend X to $X' \varepsilon C^r_G(TM)$ & conversely any such X' defines a vectorfield of the above form. Consequently,

$ev: C^r_G(TM) \times S \longrightarrow T_S H$ is transversal to points. Thus we may apply Thom's Transversality density theorem, as stated in Abraham 1, page 48, with $W = V_S TH$ to give:

Proposition 29

If $s \varepsilon C^r_G(TM)$ we may arbitrarily C^r approximate s by s' s.t.:

$$(s'|S) \bar{\pitchfork} V_S TH \subset T_S H.$$

We see that $(s'|S)^{-1}(V_S TH)$ has codimension 0. Thus we have a finite set of points $y_i \varepsilon \bar{S}'$ s.t. $s'(y_i) \varepsilon V_S TH, i=1,..,P$

This is an open condition, since $\bar{S}'$ is compact, therefore we have an open nbd. W of s' s.t. if $k \varepsilon W$ the above holds with the same value of P.

With the above notation, let C_i be the integral curve of s' through y_i. Now $C_i \subset G(y_i)$, either C_i is a closed orbit of type B, or not.

Thus there exists a finite subset $\{y_i'\}_{i \varepsilon P'}$ of $\{y_i\}_{i=1,..,P}$ s.t. $P' \subset P$, & the orbits of s' through y_i' are closed. We may then apply Theorem 18 to to each $G(C_i)$ to produce a new vectorfield $s^* \varepsilon C^r_G(TM) \cap W$, s.t. each C_i is a 2-generic closed orbit for s^* & these are the only closed orbits (mod G) of s^* meeting $\bar{S}'$, by the definition of W.

We may also apply Proposition 28 to insist that each C_i is a 2' generic closed orbit for $s^*, i \varepsilon P'' \subset P'$.

Thus we have proved:

Proposition 30

If q is a closed orbit of type B of $s \in C_G^r(TM)$ & $\text{type}(q)=i$, we may choose a G-tubular nbd. pair (U,V) of $G(q)$, a corresponding Poincare disc pair (D_1,D_2) for q at x & a C^r equivariant perturbation s^* of s with $s^*=s$ outside U s.t.; for s^*:

1. Every closed orbit of type B which meets $\bar{V}\cap M_i$ is 2-generic.
2. We may further insist that every orbit of 1. is 2^*-generic.
3. There exists a nbd. $N_{(U,V)}$ of s in $C_G^r(TM)$ s.t. the above statements hold for $s' \in N_{(U,V)}$. The point here is that our tubular nbd. pair (U,V) & disc pair (D_1,D_2) are fixed.

Remark:

In fact, with the above notation, we can also assert that $\exists s^*$ & $N_{(U,V)}$ s.t. if $j \in N_{(U,V)}$ then every orbit of q of type B which meets $\bar{V}\cap M_i$ is 2-generic (but not necessarily 2^*-generic). To see this we make all sets $G(y_i)$, $i \in P$, into 2-generic closed orbit families of type B-of perhaps very high period-then the $G(y_i)$ are normally hyperbolic sets for s^* &, by a result of P-H-S this is an open condition in $C^r(TM)$. This gives us the result.

Case A

Let $s \in C_G^r(TM)$ & let q be a closed orbit of type A. Let f_s denote the generalised Poincare map for q. We have the following trivial adaptation of a well known proposition (see Peixoto 1 for background):

Proposition 31

If $s+\Delta s$ is a C^r equivariant perturbation of s, then $s+\Delta s$ has a C^r equivariant generalised Poincaré map $f_{s+\Delta s}$ & $f_{s+\Delta s}$ is C^r close to f_s.

Our aim is to prove a converse result to Proposition 31, corresponding to Proposition 30. We first prove a Lemma:

Lemma 9

If M & N are G-manifolds & $f \in \mathfrak{E}^r_G(M,N)$, then there exists a nbd. U_f of f in $\mathfrak{E}^r_G(M,N)$ s.t. if $g \in U_f$, g is C^r equivariantly isotopic to f, i.e. $\exists$ a C^r map $H: M \times I \longrightarrow N$ s.t.:

1. $H_0=f, H_1=g$, where $H_s(x)=H(x,s)$
2. $H_t \in \mathfrak{E}^r_G(M,N)$, $t \in I$.
3. We may insist that if $fx=gx$, $H_t(x)=fx$, $\forall t \in I$.
4. We may insist that $H_t=f$, for t in some nbd. of 0 in I & $H_t=g$ for t in some nbd. of 1 in I.
5. If g is C^r close to f, we may insist that H_t is C^r close to f, $t \in I$.
6. If $D_2H(x,t)$ is the time derivative of H, at (x,t), we may insist that $D_2H(x,t)$ is a $\underline{C}^r$ function of x.

Proof

The above is well known when $G=\mathrm{id}$. For the equivariant case we proceed in an analogous manner:

First consider some C^∞ function $k: I \longrightarrow I$, s.t. $k=0$ in a nbd. of 0 & 1 in a nbd. of 1. Take an equivariant spray on TN & let S & exp denote the corresponding flow & exponential map. Let T_N denote the tangent bundle projection. Define:

$$H(x,t)=T_N S(\exp_x^{-1}(gx),k(t)), t \in I.$$

Then H satisfies conditions 1 to 6, for g in some sufficiently small nbd. U_f of f.

For the significance of condition 6, see the additional remarks on page 122.

Now, with the notation of section 8, $f_s:N_r \longrightarrow N^*$. We will suppose w.l.o.g. that, if $f_s(y)=F_{T(y)}(y), y\in N_r$, then:

$$\overline{\bigcup_{y\in N_r} F([0,T(y)],y)}\subset N \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;1.$$

We will also suppose that for some $t\in R, r>t>0$, we have:

$$\overline{\bigcup_{y\in N_t} F([0,T(y)],y)}\subset G(N_r) \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;2.$$

We note that 1. & 2. are open conditions on s, in $C^r_G(TM)$. Let $x\in q$, then if $G_q/G_x \cong C^k$, we have:

$$N^*\cap q=\{x,hx,\dots,h^{k-1}x\}, \text{for some } h\in G_q.$$

Since s is transversal to N_r it is easy to see that f_s factors as:

$$f_s=\psi\circ\dots\circ\psi=\psi^k, \text{where } \psi:N_r\longrightarrow N^* \;\&\; \psi^s(x)=h^s(x), \text{i.e.}$$

the s-orbit through $y\in N_r$ meets N^* precisely $k+1$ times in the time interval $[0,T(y)]$, we use condition 1. here.

We assert ψ is equivariant: This follows in exactly the same way as we showed that f_s was equivariant & we get $\psi(y)=F_{T^*(y)}(y)$, where $T^*:N_r\longrightarrow R$ is equivariant. Now let:

$L=\{z\in N^*:G_z=G_x\}, L_r=\{z\in N_r:G_z=G_x\}, N_r^i=N_r\cap M_i, N^i=N^*\cap M_i.$

$S^y=\{z\in N^*_y:G_z=G_x\}, S^y_r=\{z\in(N_r)_y:G_z=G_x\}.$ $(y\in q)$

We note that $L_{(r)}$ & $S^y_{(r)}$ are trivial G_x manifolds & L is an $N(G_x)$ manifold. We distinguish two cases:

<u>Case 1</u>

$\mathrm{Dim}(L)=\dim(S^y)$-this corresponds to $\dim N(G_x)=\dim(G_x)$.

We note that $hS^x=S^{hx}$, since S^x is a q-slice. We let $\psi|S_r^{h^i x}=\Phi_i$, then $\Phi_o:S_r^x \longrightarrow S^{hx}$ & we define

$$\Phi_h=h^{-1}\cdot\Phi_o:S_r^x \longrightarrow S^x.$$

Now $(f_s|S_r^x)=\Phi_{k-1}\cdot\ .\ .\ \cdot\Phi_o$. Using the equivariance of ψ, $\Phi_i=h^i\cdot\Phi_o\cdot h^{-i}$. Thus:

$$f_s|S_r^x=(h^{k-1}\Phi_o h^{-(k-1)})\ldots.(h\Phi_o h^{-1})\Phi_o,$$
$$=(h^{k-1}\Phi_o)(h^{-1}\Phi_o)^{k-1}=h^k(h^{-1}\Phi_o)^k.$$

Thus $\underline{(f_s|S_r^x)=h^k\Phi_h^k.}$

Now $\Phi_h:S_r^x\longrightarrow S^x$, by the Kupka-Smale density theorem for diffeomorphisms we may C^r approximate Φ_h by Φ_h', s.t. $\Phi_h'^k|\bar{S}^x_{t/2}$ is generic & $\Phi_h'=\Phi_h$ outside of $S^x_{t'}$, $t'<t$.

Define $\psi':S_r^x\longrightarrow S^{hx}$, by $\psi'(y)=h\cdot\Phi_h'(y)$, we extend ψ' to $\bar{\Psi}:N_r^i\longrightarrow N^i$ by:

$$\bar{\Psi}(gy)=g\,\psi'(y);\,y\in S_r^x,g\in G.$$

We assert that $\bar{\Psi}$ is a well defined C^r equivariant map. This is so, for suppose $z=gy=ky'$, y & $y'\in S_r^x$, g & $k\in G$. We see that $y=y'$-since $G(z)$ meets S_r^x in a unique point, hence $g^{-1}k\in G_x$. But $g\,\psi'(y)=k\,\psi'(y)$ iff $k^{-1}g\in G_{\psi'(y)}$, but since $\psi'(y)\in S^{hx}\subset L$, this is so, therefore $\bar{\Psi}$ is well defined. The rest of the assertion is trivial. Thus:

$$\bar{\Psi}:N_r^i\longrightarrow N^i\ \&\ \bar{\Psi}|(N_r^i-N_{t'}^i)=\psi, \&\ \ \psi \text{ is } C^r \text{ close to } \bar{\Psi} \text{ on } N_r^i.$$

Now let $y\in S_t^x$:

$$\bar{\Psi}^2(y)=\bar{\Psi}(\bar{\Psi}(y))$$
$$=\bar{\Psi}(h\Phi_h'(y))\text{-by definition of }\psi',$$
$$=h\bar{\Psi}(\Phi_h'(y))$$
$$=h(h\Phi_h'^2(y))\text{-since }\Phi_h'(y)\in S_r^x$$
$$=h^2\Phi_h'^2(y).$$

Proceeding inductively we show:

$\underline{\bar{\Psi}^k(y)=h^k\phi_h'^k(y)}$

Now we note that $\bar{\Psi}^k(y)=y$ iff $h^k\phi_h'^k(y)=y$

iff $\phi_h'^k(y)=y$, since $h^k=id$ on S^x.

Thus $Fix(\bar{\Psi}^k|\bar{S}^x_{t/2})=Fix(\phi_h'^k|\bar{S}^x_{t/2})=\{y_i\}_{i\in Q}$ say, where Q is finite. Now the condition '$\phi_h'^k|\bar{S}^x_{t/2}$ is generic' is an open condition. Thus, if we consider the set E of C^r equivariant embeddings of N^i_r into N^i, with the induced C^r topology, with the additional property that, if $g\in E$ then $g=\Psi|(N^i_r-N^i_{t*})$ for some $t*<t$, then we have a nbd. U of $\bar{\Psi}$ in E, s.t. if $j\in U$, $card(Fix(j^k|\bar{S}^x_{t/2})=card(Fix(\phi_h'^k|\bar{S}^x_{t/2})$. In fact the 'Fix' sets are clearly isotopic by an isotopy depending continuously on j.

Case 2

$Dim(L)>dim(S^x)$. This corresponds to the case where $dim(N(G_x))>dimG_x$. The treatment of this case paralles that of case 1, though one has to proceed carefully.

We have $f_s=\psi^k$. Consider $\psi_o=\psi|S^x_r:S^x_r\longrightarrow L$. Although $\psi_o(S^x_r)$ is not, in general, contained in S^{hx}, we still have $\psi_o(x)=hx$.

Consider $\Phi=ph^{-1}\psi_o:S^x_r\longrightarrow S^x$, where p is the fiber bundle map of $G(S^x)$ onto S^x (just project down G-orbits & note that, if $y\in S^x$, $G(y)$ meets S^x in a unique point).

The first point to note is that Φ is an embedding. To see this, suppose $\psi_o(y)=g\psi_o(z)$, y & $z\in S^x_r$. Then, since the flow is equivariant, this implies that $y=gz$ & so $y=z$, since S^x_r is a q-slice. Thus Φ is an embedding.

Next we note that we may, w.l.o.g., define Φ by $\Phi(y)=K(y)\cdot\psi(y)$, where $K:S^x_r\longrightarrow G$ is a C^r map. To see this we construct

$K':S_r^x \longrightarrow G$, s.t. $\bar{\Phi}(y)=K'(y).h^{-1}\psi_o$, by using a cross section of G_x: we omit details. Set $K(y)=K'(y).h^{-1}$ & also define $J:S_r^x \longrightarrow G$, by $J(y)=(K(y))^{-1}$. Then J is also a C^r map.

Now let us perturb $\bar{\Phi}$ to ϕ, using the Kupka-Smale density theorem for diffeomorphisms so that $\phi^k|\bar{S}^x_{t/2}$ is generic & $\phi=\bar{\Phi}$ outside $S^x_{t'}$, for some $t'<t$. We define, for $y\in S^x_r$,

$$\underline{\bar{\Psi}(y)=J(y)\cdot\phi(y).}$$

Thus $\bar{\Psi}:S_r^x \longrightarrow L$. We extend $\bar{\Psi}$ to N_r^i:

$$\bar{\Psi}(gy)=g\bar{\Psi}(y);\ y\in S_r^x, g\in G.$$

As in case 1, $\bar{\Psi}$ is a well defined C^r equivariant map.

Fix $y\in S_t^x$. We write $J(\phi^i(y))=g_i\in G$, for $0\leq i\leq k-1$. We note that all these g_i are defined by condition 2. on t, at least if ϕ is sufficiently C^r close to $\bar{\Phi}$-i.e. $\bar{\Phi}^i(S_t^x)\subset S_r^x$.

Then we have:

$$\bar{\Psi}^2(y)=\bar{\Psi}(g_o\cdot\phi(y)).$$
$$=g_o\bar{\Psi}(\phi(y))\text{-using the equivariance of }\bar{\Psi}.$$
$$=g_o g_1\phi^2(y).$$

Proceeding inductively we may show:

$$\bar{\Psi}^k(y)=g_o\ldots g_{k-1}\phi^k(y).$$

Now suppose $\bar{\Psi}^k(y)=y$, then $y=(g_o\ldots g_{k-1})\phi^k(y)$, but $\phi^k(y)\in S_r^x$ & so $g_o\ldots g_{k-1}\in G_x$. Thus we have:

$\mathrm{Fix}(\phi^k|\bar{S}^x_{t/2})\supset \mathrm{Fix}(\bar{\Psi}^k|\bar{S}^x_{t/2})=\{y_i\}_{i\in Q}$, & these sets are all finite.

1 Now we have:

1. $\bar{\Psi}=\psi$ outside of N_t^i, & $\bar{\Psi}$ is C^r close to ψ.

& we may easily show that:

2. With the definition of E as in case 1, we have a nbd. $U_{\bar{\Psi}}$

of $\bar{\Psi}$ in Ξ,s.t. if $j\varepsilon U_{\bar{\Psi}}$,card(Fix($\bar{\Psi}^k|\bar{S}^x_{t/2}$))>card(Fix($j^k|\bar{S}^x_{t/2}$)) & in fact Fix($j^k|\bar{S}^x_{t/2}$) is isotopic to a subset of Fix($\bar{\Psi}^k|\bar{S}^x_{t/2}$) by an isotopy depending continuously on j.

In either case 1 or 2, we use Lemma 9 to construct an equivariant C^{r+1}-isotopy between id & $\psi^{-1}\cdot\bar{\Psi}|N^i_r$, noting that, if H denotes the isotopy, H_t=id outside of N^i_t. Thus:

$$H:N^i_r \times [0,c] \longrightarrow N^i.$$

We suppose $H_t=\psi^{-1}\cdot\bar{\Psi}$ for t in some nbd. $[d,c]$ of c in $[0,c]$, where $d<\min_{y\varepsilon L_r}(T^*(y))<\max_{y\varepsilon L_r}(T^*(y))<c$, & H_t=id on a nbd. of 0.

Then we define $\bar{F}:D\subset N^i_r \times R \longrightarrow N^i$, where $D=\bigcup_{y\varepsilon N^i_r}([0,T^*(y)] \times y)$

by $\bar{F}_t(y)=F_t(H_t(y))$, where F_t is the flow of s.

We note that:

$$F_{T^*(y)}(y)=F_{T^*(y)}(H_{T^*(y)}(y))=\psi(\psi^{-1}\cdot\bar{\Psi}(y))=\bar{\Psi}(y).$$

Provided that $\psi^{-1}\cdot\bar{\Psi}$ is sufficiently C^r close to the identity (& hence H_t is close to the identity) F_t is close to $\bar{F}_t$ & (openness of embeddings) $\bar{F}_t$ defines a set of integral curves for an equivariant vectorfield $\bar{s}$ on $\bar{F}(D)$ which is C^r close to s. By our choice of t & r this extends to an equivariant vectorfield s* on $(M-N)\cup N_i$, where $N_i=N\cap M_i$, with $s=s^*$ on $M-N_r-(N-N_i)$.

Finally we extend s* to s' on M, keeping s' C^r close to s*. To do this (uniformly) we take a finite cover of N_i by slices S_j, i.e. $G(S_j)$ cover $\bar{N}$, & then extend on each S_j, using the linear structure of the slice.

Then s' has a Poincare map $f_{s'}:N_r \longrightarrow N^*$, with $f_{s'}|N^i_r=\bar{\psi}^k$.

The above is only a sketch of the constuction of such a perturbation s' of s, with the required condition on the Poincare map. The constuction is well known in the G=id case.

Thus we have constucted a G tubular nbd. pair (U,V) of G(q) & a corresponding generalised Poincare section pair (D_1,D_2), $D_1=U|G(x)$, $D_2=V|G(x)$, s.t. if we restrict attention to $V_i=V\cap M_i$ we have only a finite number of closed orbits in V_i (mod G!) for s' corresponding to fixed points of $f_{s'}$ & this property is stable under equivariant perturbation in the above sense. Thus, using Theorem 18, we may perturb s' to s" $\underline{C}^r$ equivariantly s.t. these closed orbits are 2-generic closed orbits of s" & no new fixed points of $f_{s''}$ are introduced on V_i. Further, if $\dim N(G_x) > \dim G_x$ & c is such a closed orbit we may perturb s" to $s^!$, using a perturbation lying in G(c), so that $f_{s!}$ has no fixed points on V_i. More precisely: Restricting attention to G(c), suppose $G_c/G_z \cong C^1$, $z\varepsilon c$. Choose $g'\varepsilon N(G_x)$, close to the identity, s.t. g' is not periodic (note that $\dim(N(G_x))\geqslant 1$). Then if f denotes the generalised Poincare map for c, $f=X^1$. We define $\bar{X}$ by $\bar{X}(z)=g'X(z)$ & extend X equivariantly (note that $\bar{X}$ is defined on a finite set). Then $\bar{X}^1(z)=g'^1X^1(z)=g'^1z\neq z$. We then construct the equivariant flow on G(c) with Poincare map $\bar{X}^1$ & then extend this to the whole of M, so as to be C^r close to the original flow & equal to the original flow outside of some G-tubular nbd. of G(c) disjoint from G-tubular nbds of the remaining closed orbits in V_i, corresponding to fixed points of s".

Thus we have proved:

Proposition 32

If q is a closed orbit of type A of $s\varepsilon C^r_G(TM)$, then $\exists$a G-tubular nbd. triple (U,V,W) of $G(q)$ & a corresponding generalised Poincare section triple (D_1,D_2,D_3) s.t.:

1. We may C^r equivariantly perturb s to s', with $s'=s$ outside U, s.t. the generalised Poincare map of s', $f_{s'}$, is 1-generic on $D_2\cap M_i$. Further, by choice of W, we may insist that closed orbits of s', corresponding to fixed points of $f_{s'}$ & lying in M_i, which meet $\overline{W}_i$, lie inside V_i: Thus all closed orbits of s', corresponding to fixed points of $f_{s'}$, which meet $\overline{W}_i$, are 2-generic & lie in V_i.

2. If $\dim(N(G_x))>\dim G_x$, we may insist that s' has no closed orbits meeting $\overline{W}_i$, corresponding to fixed points of $f_{s'}$, i.e. we may suppose s' is 2*-generic on $\overline{W}_i$.

3. For possibly smaller W, there exists an open nbd. $N_{(U,V,W)}$ of s in $C^r_G(TM)$ s.t. the above statements are true for $j\varepsilon N_{(U,V,W)}$, with the same tubular nbd. triple (U,V,W).

We now give a definition:

Definition 31

Let $\mathfrak{K}^2_G(M;r)=\{s\varepsilon C^r_G(TM)$: s is 1-generic & all closed orbits of s are 2-generic$\}$.

Let $\mathfrak{K}^{2*}_G(M;r)=\{s\varepsilon C^r_G(TM)$: s is 1*-generic & all closed orbits of s are 2*-generic$\}$.

It is our intention to show that the above sets form a residual subset of $C^r_G(TM)$. Proposition 30 & Proposition 32 are two of the three key lemmas towards this end. The remaining lemma we need is an equivariant version of Hartman's theorem. As we do not strictly need the equivariance we could go straight to the proof (page 147ff) however, we will first develop some stable manifold theory, which we use, amongst other things, to prove an equivariant version of Hartman's Theorem.

Remark

We make one or two additional remarks concering the proofs of the preceeding propositions.

Given a C^r flow $F:M \times R \longrightarrow M$ it does not follow that F defines a C^r vectorfield on M, in general F defines only a C^{r-1} vectorfield on M. It will define a C^r vectorfield if $DF_x(0,1)$ is a C^r function of x. In our construction of a perturbed flow in Proposition 32, we considered an expression of the form $F_t(H_t(x))$, where F was the original flow on M & H was a C^r isotopy of the type defined in Lemma 9. It is because of condition 6 in Lemma 9 that we can use this expression to define a C^r vectorfield, rather than just a C^{r-1} vectorfield.

It is perhaps also worth remarking that, although C^r flows pull back to C^r flows by using C^r immersions, the corresponding C^r vectorfield pulls back to a C^{r-1} vectorfield. In Abraham 1, it is in several places assumed that C^r vectorfields do pull back by means of C^r maps to C^r vectorfields.

11. Stable Manifold Theory for Equivariant Diffeomorphisms & Vectorfields

In this section we construct (global) stable & unstable manifolds for generic critical sets of equivariant vectorfields & diffeomorphisms & obtain a 'nice' representation for these manifolds. Most of the results in this section are a straightforward consequence of results in P-H-S 1 adapted to the equivariant case. This is a consequence of the fact that the contractions defined in P-H-S on the appropriate spaces of Lipschitz functions are (of course) contractions on the closed subspaces of equivariant Lipschitz functions & hence the resulting fixed points correspond to equivariant functions. In the sequel we will not need this remark as the chacterisation of the various objects studied will always give equivariance trivially. In this section & section 12 we assume familiarity with P-H-S 1.

We divide this section into two parts: The first is on invariant manifolds for diffeomorphisms the second for invariant manifolds for vectorfields.

Invariant manifolds for diffeomorphisms.

With the notation of Definition 14, page 46, we have:

Theorem 19

If f is G,r-normally hyperbolic at V, $r \geq 1$, then there exist C^r, locally f-invariant G-invariant submanifolds passing through V, $W^u_{loc}(V)$, $W^s_{loc}(V)$, tangent to $N^u \oplus TV$ & $N^s \oplus TV$ respectively.

Proof

Everything but the G-invariance is in P-H-S 1. The G-invariance will follow from the chacterisation of these manifolds in Theorem 21.

Theorem 20

$W^s_{loc}(V)$ has a locally f & G invariant fibration, whose fiber through $p\varepsilon V$, $W^{ss}_{loc}(p)$ is tangent to the plane $N^s(p)$ at V. It is regular of class $C^{o;r}$. Similarly for $W^u_{loc}(V)$.

Proof

Again all is in P-H-S, except the equivariance of the fibration which follows from theorem 21.

Theorem 21

Locally, $W^u_{loc}(V)$, $W^s_{loc}(V)$ & their G & f invariant fibrations are unique. They are characterised by:

1. $W^s_{loc}(V)$=all points tending to V, always in a given G-invariant nbd. of V, under positive iterates of f.
2. $W^u_{loc}(V)=W^s_{loc}(V)$ for f^{-1}.
3. $W^{ss}_{loc}(p)$=all points of $W^s_{loc}(V)$, more sharply asymptotic with p, than $m(Tf^n|TV)$ is with 0, as $n\longrightarrow\infty$
4. $W^{uu}_{loc}(p)=W^{ss}_{loc}(p)$, for f^{-1}

Now suppose x is a 1-generic fixed point for f, then with V=G(x), we may define 'the local stable manifold of G(x)' by:

$$W^s_{loc}(G(x))=\{z\varepsilon U:U \text{ is an open } G\text{-invariant nbd. of } G(x) \;\& \; f^n z\longrightarrow G(x) \text{ as } n\longrightarrow\infty, f^i(z)\varepsilon U, i\varepsilon Z^+\}.$$

Then, provided U is small enough, $W^s_{loc}(G(x))$ is a G-invariant C^r submanifold of M, fibrated by $W^{ss}_{loc}(gx)$, $g\varepsilon G$, in an equivariant way. Here, we have:

$$W^{ss}_{loc}(gx)=\{z\varepsilon U:F^n(z)\longrightarrow gx \text{ as } n\longrightarrow\infty \ \& \ f^i z\varepsilon U, i\varepsilon Z^+\}.$$

We note that, given U,

$W^{ss}_{loc}(gx)=gW^{ss}_{loc}(x)$, &, as a consequence, we may regard $W^s_{loc}(G(x))$ as a C^r fiber bundle over $G(x)$. Similarly for $W^u_{loc}(G(x))$.

Now suppose $f^p(x)=x$ (where p is the prime period of x) & f^p is 1-generic at x. Let us define:

$$G(x)^p=\bigcup_{0\leq i\leq p-1} f^i(G(x)).$$

We note that $G(x)^p$ is a finite collection of G-orbits. We define $W^s_{loc}(G(x)^p)$ using f^p, similarly for $W^u_{loc}(G(x)^p)$. If $z\varepsilon G(x)^p$, we write $W^{ss}_{loc}(z^p)$ to mean $W^{ss}_{loc}(z)$ constructed using f^p. Again $W^s_{loc}(G(x)^p)$ is a C^r fiber bundle over $G(x)^p$. Similarly for $W^{uu}_{loc}(z^p)$.

Finally, before defining the global stable & unstable manifolds, we note the following theorem of P-H-S 1, adapted to the G-category:

Theorem 22

If $f'\varepsilon \mathfrak{Diff}^r_G(M)$ is C^r close to f & f is G,r-normally hyperbolic at V, then f' is G,r-normally hyperbolic at some C^r f' invariant G-submanifold V' near V.

Theorem 22 has application to generic critical sets. Thus if G(x) is a 1-generic fixed set for f, then if f' is a perturbation of f s.t. we lose the fixed point set G(x) of f, nevertheless we still have an x' near x s.t. G(x') is G & f'-invariant & f' is G,r normally hyperbolic on G(x'). Thus x' might be a point of high period, but it will still be generic.

Also it is a consequence of this result that type A generic closed orbits do not perturb into type B generic closed orbits & conversely. This can also be seen without using Theorem 22, by elementary considerations.

Now let x be a 1-generic fixed point of f^p, p prime period of x, we define the following subsets of M:

$W^s(G(x)^p)=\{z\varepsilon M: f^{pn}z \longrightarrow z \text{ as } n \longrightarrow \infty\}$.

$W^u(G(x)^p)=\{z\varepsilon M: f^{pn}z \longrightarrow z \text{ as } n \longrightarrow -\infty\}$.

$W^s(y^p)=\{z\varepsilon M: f^{pn}z \longrightarrow y \text{ as } n \longrightarrow \infty\}$, $y\varepsilon G(x)^p$

$W^u(y^p)=\{z\varepsilon M: f^{pn}z \longrightarrow y \text{ as } n \longrightarrow -\infty\}$, $y\varepsilon G(x)^p$.

We note that for $y\varepsilon G(x)^p$, $W^s(y^p)$ is a G_y-invariant subset of M (in general $G_y \neq id$ on this set). Also we see that $W^s(G(x)^p)=\bigcup_{g\varepsilon G} W^s(gx_i)$ where x_i is a finite set of points, one from each G-orbit in $G(x)^p$. Similarly for the unstable sets.

We let $T_{G(x)^p}M=T(G(x)^p)\oplus N^s\oplus N^u$ denote the splitting of $T_{G(x)^p}M$ induced by f^p. Since $G(x)^p$ is a C^∞ manifold, it is easy to see that N^u & N^s are both C^∞ G-vector bundles over $G(x)^p$.

Before proving our main result on invariant manifolds for generic fixed sets, we state & prove a lemma about the definition of G,r-normal hyperbolicity.

Lemma 10

Recalling definition 13, page 46, on G,r-normal hyperbolicity, we may assume that the Riemannian metric is equivariant.

Proof

If k was the Riemannian metric of Definition 13, we define a new metric $k'=Av(k)$. Then if $\| \|_G$ denotes the new equivariant norm, we have:

$$\|v\|_G=\int_G\|gv\|dg. \quad v\in TM.$$

We assert that this norm still works in definition 13. To see this consider, for $v\in N_x^s, x\in V$:

$$\|Tf_x(v)\|_G=\int_G\|gTf_x(v)\|dg$$

$$=\int_G\|Tf_x(gv)\|dg \text{ -by the equivariance of f.}$$

$$\leq\sup_{x\in V}\|Tf_x|N_x^s\|\int_G\|gv\|dg \text{-since the splitting is G-invariant-Proposition 9.}$$

$$=\sup_{x\ V}\|Tf_x|N_x^s\|\|v\|_G.$$

Thus: $\sup_{x\in V}\|Tf_x|N^s\|_G\leq\sup\|Tf_x|N_x^s\|$.

Similarly: $\inf_{x\in V}(m_G(Tf|TV))\geq\inf_{x\in V}(m(Tf|TV))$, where m_G denotes that we have calculated w.r.t. the G-norm. Thus:

$$\sup\|Tf|N^s\|_G<\inf m_G(Tf|TV).$$

The other relations follow similarly.

Theorem 23

With the above notation, $W^s(G(x)^p)$ is the image of an injective C^r equivariant immersion I^s of N^s into M & further, $I^s(N_z^s)=W^s(z^p), z\in G(x)^p$. Similarly for the unstable manifold.

Proof

We divide the proof into two parts: First a local result which follows from P-H-S $\underline{1}$ &,secondly,a global result.

We recall from P-H-S $\underline{1}$ that,since $f|W^s_{loc}(G(x)^p)$ is a d-pseudo hyperbolic operator ($d<1$) on the d-pseudo hyperbolic set $G(x)^p$,we may regard the family $\{W^{ss}_{loc}(z^p)\}_{z\in G(x)^p}$ as a regular family of C^r submanifolds of M.In fact it is shown in P-H-S $\underline{1}$ that we have a regular family of submanifolds $\{\overline{W}_1(z^p)\}_{z\in G(x)^p}$ of $T_{G(x)^p}M$,where $\overline{W}_1(z^p)\subset T_zM$ as a C^r submanifold tangent to N^s_z at the origin & we have:

$$\exp_z(\overline{W}_1(z^p))=W^{ss}_{loc}(z^p) \quad \ldots\ldots\ldots 1.$$

By choosing an equivariant Riemannian metric on M,we can ensure that 'exp' is equivariant & hence that $\{\overline{W}_1(z^p)\}_{z\in G(x)^p}$ forms a G-invariant family of C^r manifolds. Let $\overline{W}^s_{loc}(G(x)^p)=\bigcup_{z\in G(x)^p}\overline{W}_1(z^p)$.We assert that $\overline{W}^s_{loc}(G(x)^p)$ is contained in $T_{G(x)^p}M$ as a C^r submanifold.This is clear,since $G(x)^p$ is a finite union of G-orbits & thus $\overline{W}^s_{loc}(G(x)^p)=\bigcup_{i\in Q}G(\overline{W}_1(z^p_i))$,where Q is finite.

Let $p_s:T_{G(x)^p}M=N^s\oplus(N^u\oplus TG(x)^p)\longrightarrow N^s$ be the projection map.Then for some sufficiently small nbd. of the zero section of $T_{G(x)^p}M$,$p_s|\overline{W}^s_{loc}(G(x)^p)$ is a C^r-embedding onto an open nbd. of the zero section in N^s.We denote the local inverse by p_s^{-1}.

Thus we may define:

$I^s_{loc}:N^s_{\delta'}\longrightarrow W^s_{loc}(G(x)^p)\subset M$,where $N^s_{\delta'}$ is some suitable open disc bundle of N^s,radius δ' &

$$I^s_{loc}(e_z)=\exp_z . p_s^{-1}(e_z), e_z \varepsilon (N^s_{\delta'})_z .$$

I^s_{loc} is thus a C^r equivariant embedding of N^s, into M,with $I^s_{loc}((N^s_{\delta'})_z)=W^{ss}_{loc}(z^p)\subset M$. This concludes part 1 of the proof.

Recall from Palis 1 the definition of fundamental domain. Since $Tf^\rho|N^s=N_sf^\rho$ is a contraction (recall lemma 10) we have a fundamental domain D_s for N_sf^ρ in N^s ,we may also insist that D_s is G-invariant.(i.e. we just take B_s,the unit disc bundle of N^s & define $D_s=B_s-N_sf^\rho(B_s)$).

W.l.o.g. let us assume $D_s\subset N^s_{\delta'/2}$. We define I^s as follows:

If $e\varepsilon N^s$,there exists a unique $n\varepsilon Z$,s.t. $(N_sf)^{\rho n}e\varepsilon D_s$, define:

$$I^s(e)=f^{-n\rho}I^s_{loc}(N_sf)^{\rho n}e.$$

It is then easy to check that I^s has the required properties.

We will also obtain a theorem about how I^s varies with f,we will leave this,however,to the end of the section on equivariant vectorfields.

Invariant manifolds for equivariant vectorfields

Theorems 19,20,21 generalise to flows immediately & we have:

Theorem 24

If X is G,r-normally hyperbolic at V,r$\geqslant$1,then:

1. There exist C^r locally G,X-invariant submanifolds passing through V,$W^u_{loc}(V)$,$W^s_{loc}(V)$ tangent to $N^u\oplus TV$ & $N^s\oplus TV$ respectively.

2. $W^s_{loc}(V)$ has a locally G,X-invariant fibration whose fiber through $p\varepsilon V$,$W^{ss}_{loc}(p)$,is tangent to the plane $N^s(p)$ at V.It is regular of class $C^{o,r}$.Similarly for $W^u_{loc}(V)$.

3. Locally $W^u_{loc}(V)$,$W^s_{loc}(V)$ & their G,X invariant fibratiions are unique.They are characterised by:

A. $W^s_{loc}(V)$=all points tending to V,always in a given nbd. of V,under positive iterates of F_t,$t>0$.

B. $W^u_{loc}(V)=W^s_{loc}(V)$ for F_{-t}.

C. $W^{ss}_{loc}(p)$=all points of $W^s_{loc}(V)$,more sharply asymptotic with p,than $m(TF^n_t|TV)$ is with 0,as $n\longrightarrow\infty$, $t>0$.

D. $W^{uu}_{loc}(p)=W^{ss}_{loc}(p)$,for F_{-t}.

4. For $t>0$,all the above manifolds are independent of t.

Thus if q is a generic closed orbit of $X\varepsilon C^r_G(TM)$,we have local stable & unstable manifolds for G(q).We also note that $W^s_{loc}(q)=\bigcup_{x\varepsilon q} W^{ss}_{loc}(x)$ is a C^r submanifold:This is a consequence of the fact that the flow of X is transitive on q,we omit details.

Next we introduce 'fundamental equivariant domains' for invariant manifolds of equivariant flows.These are constructed for G=id in P-H-S 1.The methods used here are a simple generalisation of those in P-H-S 1:

Let $\{f_t\}$ be an equivariant C^1 flow on some G-manifold M.We say that V is a uniform attractor for this flow,if,for some compact (equivariant) nbd. K of V,we have:

$$\bigcap_{t\geq 0} K_t=V, \text{where } K_t=\{f^s(x):x\epsilon K, s\geq t\}.$$

V is,of course,G-invariant.Similarly for repellors.

We have the following 'Liapunov' theorem:

Theorem 25

If $\{f^t\}$ is a C^1 equivariant flow on a finite dimensional smooth G-manifold & if V is a uniform attractor, then there is a smooth equivariant Liapunov function for the flow at $V, H: M \longrightarrow \mathbb{R}$, s.t.:

1. $H^{-1}(0)=V$.
2. H decreases along trajectories near V, $\dot{H}(t)<0$, along the flow, near V.

Proof

The result is proved for G=id in P-H-S 1. The result depends, as in P-H-S 1, on the following lemma:

Lemma

Let $\{K_n\}$ be a family of G-invariant compact nbds of a compact G-invariant set V in $\mathbb{R}^m$, where G acts on $\mathbb{R}^m$ as a group of orthogonal transformations. We suppose $K_n \longrightarrow V$, i.e.

$$\bigcap_{k\geq 0}\left[\bigcup_{n\geq k} K_n\right]=V.$$

Let $\{\varepsilon_n\}$ be a given sequence of positive numbers. Then there exists a smooth equivariant function $h: \mathbb{R}^m \longrightarrow \mathbb{R}$, s.t.

a) $h=0$ on V, $h>0$ off V.

b) $\|h|K_n\|_r \leq \varepsilon_n$, if $r\leq n$. (Here our r-norm is supposed G-invariant).

Proof

This lemma is proved, for G=id, in P-H-S 1. That it is true in the equivariant case, follows by merely constructing $\bar{h}$ for the above conditions on K & ε, & then defining $h=Av(\bar{h})$. Since $\|Av\|_r=1$, the result follows, since the K_i are

G-invariant.

Then we take a locally finite set of slices S_{x_i} in M, which cover M, mod G. Using these we define an equivariant p.o.1. $\{j_i\}$ s.t. $\overline{\text{supp}}(j_i) \subset G(S_{x_i})$.

Using this p.o.1. we globalise the above lemma to the whole of M, measuring $\| \ \|_r$ by means of this p.o.1.

We then proceed to complete the proof of theorem 25 as in P-H-S 1.

Let $s \in C^r_G(TM)$ & suppose q is a generic closed orbit. Suppose that we have constructed $W^s_{loc}(G(q))$. We say that a C^r equivariant fundamental domain for $W^s_{loc}(G(q))$ is a C^r G-invariant submanifold D^s_q of $W^s_{loc}(G(q))$ of codimension 1, bounding a nbd. of V, & across which s points transversally inwards. We have a similar definition for $W^u_{loc}(G(q))$ & for invariant manifolds of general G,r hyperbolic sets.

The existance of equivariant C^r fundamental domains is assured by Theorem 25.

We let $T_{G(q)}M = T(G(q)) \oplus N^u \oplus N^s$ denote the splitting of $T_{G(q)}M$ induced by F_t. Here $G(q)$ is only a C^r manifold, so that N^u & N^s are only C^{r-1} bundles. We let $\bar{N}^u$ & $\bar{N}^s$ be C^r G-vector bundle approximations to N^u & N^s respectivly.

As in the section on diffeomorphisms we define:

$$W^s(G(q)) = \{z \in M : F_t z \longrightarrow G(q) \text{ as } t \longrightarrow \infty\}.$$
$$W^s(q) = \{z \in M : F_t z \longrightarrow q \text{ as } t \longrightarrow \infty\}.$$
$$W^s(x) = \{z \in M : F^n_t z \longrightarrow x \text{ as } n \longrightarrow \infty\}, x \in G(q).$$

We note that the above sets are G, G_q, G_x invariant respectively.

Similarly we define the corresponding unstable sets.

Theorem 26

With the above notation $W^s(G(q))$ is the image of a C^r injective equivariant immersion I^s of $\bar{N}^s$ into M. We have $I^s(\bar{N}^s_{gq})=gW^s(q), g\in G$.

Similarly for the unstable manifold. Also, regarding a 1-generic singular point as a generic closed orbit of prime period zero, we have a corresponding result for singular points-here we may take $\bar{N}^s=N^s, \bar{N}^u=N^u$ as $G(x)$ is C^∞.

Proof

Again we have a local result, which is a consequence of work in P-H-S, & a global result.

We may represent $W^s_{loc}(G(q))$ as the image of a regular family of C^r submanifolds $\{\bar{W}_1(z)\}_{z\in G(q)}$ of $T_{G(q)}M$. This follows from theorem 23, noting that $W^s_{loc}(G(q))$ is just the local stable manifold of the diffeomorphism F_t, for some strictly positive t. Thus we have, restricting attention to F_t,

$$\exp_z(\bar{W}_1(z))=W^{ss}_{loc}(z), z\in G(q).$$

Noting, from Theorem 24, that $W^{ss}_{loc}(z)$ is independent of $t>0$, our family $\{\bar{W}_1(z)\}$ is independent of $t>0$.

Now let $\bar{W}_1(q)=\bigcup_{z\in q}\bar{W}_1(z)$. We assert that $\bar{W}_1(q)$ is a C^r G_q-invariant submanifold of T_qM. To see this, we note that $K:\mathbf{R}\times\bar{W}_1(z)\longrightarrow T_qM;(s,y)\longmapsto \exp^{-1}_{F_s(z)}(F_s(\exp_z(y)))$ is a C^r immersion of $\mathbf{R}\times\bar{W}_1(z)$ onto $\bar{W}_1(q)$.

As an easy consequence of this, we may then show that $\bar{W}_1(G(q))=\bigcup_{g\in G} g\bar{W}_1(q)$, is a C^r submanifold of $T_{G(q)}M$.

As in Theorem 23, we may define the projection map $p_s: T_{G(q)}M=\bar{N}^s\oplus(\bar{N}^u\oplus TG(q))\longrightarrow\bar{N}^s$, & in some sufficiently small nbd. of the zero section of $T_{G(q)}M$, $p_s|\bar{W}_1(G(q))$ is an embedding of $\bar{W}_1(G(q))$ into $\bar{N}^s$, & clearly we may, for some $\bar{\delta}>0$, define $I^s_{loc}:\bar{N}^s_{\bar{\delta}}\longrightarrow M$, by:

$$I^s_{loc}(v)=\exp_{T_M}(v)\cdot p_s^{-1}(v).$$

I^s_{loc} is clearly a C^r equivariant embedding, & this completes the local part of the proof.

Let $h:\mathbb{R}^+\longrightarrow(0,\delta)$ be a C^∞ diffeomorphism, s.t.

1. $h(t)=t$, $0\leqslant t\leqslant\delta/2$, $\delta<\bar{\delta}$
2. We define, in addition, $h(0)=0$.

Define a C^∞ equivariant compression, H, of $\bar{N}^s$ by:

$$H(e)=\frac{h(\|e\|)}{\|e\|}e,\ e\neq 0$$
$$=0,\ e=0.$$

Now let us consider the flow $\{F_t\}$ of s, restricted to $W^s_{loc}(G(q))$ - we note that it is not defined on the whole of R. We pull back this flow to a flow $\{\bar{F}_t\}$ on $\bar{N}^s$:

$\bar{F}_t(e)=H^{-1}I^{s-1}_{loc}(F_t(I.H(e)))$, whenever this expression is defined.

Now we note that, if $\|e\|\leqslant\delta/2$, then

$$\bar{F}_t(e)=I^{-1}.F_t(Ie) \quad \ldots\ldots\ldots\ldots 1.$$

Clearly, $\bar{F}_t$ is defined for $t\geqslant 0$, &, if $\|e\|\leqslant\delta/2$, then, if we look at $I(\bar{N}^s_\delta)$, $\bar{F}_t$ is certainly defined for negative values of t, which do not move $F_t I(e)$ outside of $I(\bar{N}^s_{\delta/2})$, i.e. given $\|e\|\leq\delta/2, \exists t\in R$, s.t. $\|\bar{F}_t(e)\|=c$, where $c\leq\delta/2$, $e\neq 0$.

Now let us choose a fundamental domain D^s_q for F_t on $W^s_{loc}(G(q))$ s.t. $D^s_q \subset I(\bar{N}^s_{\delta/4})$.

We lift D^s_q, by I^{-1}, to $\bar{N}^s$ & let $I^{-1}(D^s_q)=P^s_q$. P^s_q is, of course, a fundamental domain for $\bar{F}_t$.

Now suppose $e\in\bar{N}^s$, then the orbit of $\bar{F}_t$ meets P^s_q in a unique point $\bar{F}_{T(e)}(e)$. Since the C^r flow $\bar{F}_t$ is transversal to P^s_q, $T:\bar{N}^s \longrightarrow R$ is a C^r function, which is also equivariant.

Finally we define $I^s:\bar{N}^s\longrightarrow M$ by:

$$I^s(e)=F_{-T(e)}(I^s_{loc}\bar{F}_{T(e)}(e)).$$

I^s is obviously a C^r equivariant injective map onto $W^s(G(q))$. Also we note that $I^s|(\bar{N}^s_{\delta/2})=I^s_{loc}$.

We have to check that I^s is an immersion. First note that if $e\in\bar{N}^s$, then the integral curve of $\bar{F}_t$ through e meets P^s_q transversally. Thus if U is a nbd. of $F_{T(e)}(e)$ in P^s_q, $F_{-T(e)}(U)$ meets the integral curve through 0 transversally.

It is sufficient to check that this transversality is preserved in $W^s(G(q))\subset M$ under the map I^s - this is obvious since D^s_q is a fundamental domain for F_t.

We now wish to extend this result to get parametrizations of the stable & unstable manifolds of generic critical elements, for vectorfields X' near X, which depend continuously on X' in the C^r sense.

<u>Remark</u>: By a parametrization, we will always understand a map of the type constructed in Theorem 26.

For simplicity we will suppose that our generic critical elements are *-generic,i.e. they do not vanish under perturbation.We will work here just with 2*-generic closed orbits,the results & proofs are similar for 1*-generic singular sets.

Thus suppose '$G(q)$' is a family of 2*-generic closed orbits of the C^r vectorfield X.Now if X' is a C^r perturbation of X,then q perturbs to $q(X')$ near by & X' is 2*-generic on $G(q(X'))$.

We assert that there exists a nbd. U_X of X in $C^r_G(TM)$ & a map $K:U_X \longrightarrow \mathfrak{Diff}^r_G(M)$ s.t.:

1. If $Z\in U_X$,Z has a (unique) 2*-generic closed orbit $q(Z)$ near q.
2. $K(Z)q(Z)=q$.
3. K is continuous,w.r.t. the C^r topologies on range & domain.

First,as in section 9,we take a C^r G normal bundle N of $G(q)$.Let $N|q=N_q$.Then N_q is a C^r G_q vector bundle over q.N defines an associated C^r G tubular nbd. of $G(q)$ & N_q a C^r G_q submanifold of M,we still denote these sets by N & N_q respectively.

Suppose that $x\in q$ & type$(x)=i$,then consider $N^i_q=N_q\cap M_i$. Now $(N^i_q)_x$ defines a Poincare section at $x\in q$ for $X|M_i$.There certainly exists some nbd. U'_X of X in $C^r_G(TM)$ s.t. if $Z\in U'_X$ then the Poincare map of Z has a unique fixed point $x(Z)$ in $(N^i_q)_x$ which depends continuously on Z (this is essentially theorem 22 or theorem 7).Let $q(Z)$ denote the closed orbit corresponding to $x(Z)$. Letting ${}_uN^i_q$ denote the disc bundle of N^i_q of radius u, we will suppose that U'_X is chosen so that $q(Z)\subset {}_1N^i_q$, $z\in U'_X$.

Now $q(Z)$ defines a C^r-equivariant section $J(Z)$ of the bundle N^i_q: This is general & part of the proof of Thom's isotopy theorem as presented in Abraham 1. Further J depends continuously on Z. Thus:

$$J:U'_X \longrightarrow C^r_G(N^i_q), \text{is continuous.}$$

We define a C^∞ G_q-invariant map $\mathfrak{b}:N^i_q \longrightarrow R$, by insisting that:

1. $\mathfrak{b}(q)=1$.
2. $\overline{\mathrm{supp}}(\mathfrak{b})\subset {}_{\frac{1}{2}}N^i_q$.

We now define $L:U'_X \times N^i_q \longrightarrow N^i_q$ by:

$$L(Z,y)=(1-\mathfrak{b}(y))y+\mathfrak{b}(y)J(Z,py).$$

Here '+' is in the fiber of N^i_q & p is the bundle projection of N^i_q. It is clear that, for fixed Z, L is a C^r-G_q invariant map which equals the identity on some collar nbd. of the boundary of N^i_q.

Thus we have a map:

$$L^*:U'_X \longrightarrow \mathfrak{C}^r_G(N^i_q,N^i_q)^*; Z\longmapsto L_Z.$$

It is easy to check that L^* is a continuous function. Since $L^*(X)=\mathrm{id}$, we have a nbd. U''_X of X, s.t.

$$L^*(U''_X)\subset \mathfrak{D}\mathrm{iff}^r_G(N^i_q,N^i_q)^*.$$

We now wish to extend our definition of L^*, so that $L^*(Z)$ is an equivariant diffeomorphism of M.

To do this note that $N_q=N^i_q\oplus\hat{M}$, where $\hat{M}$ is the orthogonal complement of N^i_q in N_q & is G_q-invariant. We extend $L^*(Z)$ to

*Strictly: C^r maps, supported inside N^i_q, w.r.t. id, induced topology.

a C^r G_q map of N_q, as we extended it to a map of N_q^i, by means of a suitable G_q invariant C^∞ function $\mathfrak{W}: N_q \longrightarrow R$, corresponding to $\mathfrak{b}$. Then for perhaps a smaller U_X'', L^* maps continuously into the diffeomorphisms of N_q. We then extend L^* equivariantly, using G, to the whole of N, & then set $L^*(Z)=\mathrm{id}$ on $M-N$.

Setting $U_X''=U_X$ & $L^*=K$ completes the construction.

Remark: The above is essentially just Thom's Isotopy Theorem, Thom 1, page 26, for equivariant maps. We give the argument here merely to emphasise the dependence of K on Z.

Now let us consider the set of C^r equivariant flows on M: $\mathfrak{F}_G^r(M)$. We will topologise $\mathfrak{F}_G^r(M)$ as follows. Let $\{f^t\} \varepsilon \mathfrak{F}_G^r(M)$ & let U be a C^r-nbd. of $f^1 \varepsilon \mathfrak{Diff}_G^r(M)$, then we define a C^r nbd. $\tilde{U}$ of $\{f^t\}$ by insisting that $\{g^t\} \varepsilon \tilde{U}$ iff $g^1 \varepsilon U$.

Given $C_G^r(TM)$, we have a natural map $F: C_G^r(TM) \longrightarrow \mathfrak{F}_G^r(M)$, defined by sending $X' \varepsilon C_G^r(TM)$ to $\{F_t^{X'}\}$. F is continuous, w.r.t. the above defined topology on $\mathfrak{F}_G^r(M)$ - this is a trivial consequence of, for example, 'the parametrised flow theorem' in Abraham 1.

We have, corresponding to the open nbd. U_X of X constructed above, a subset $V_X = F(U_X)$ of $\mathfrak{F}_G^r(M)$. We consider some nbd. W_X of $F(X)$ in $\mathfrak{F}_G^r(M)$ & set $\mathfrak{V}_X = W_X \cap V_X$ & let $\mathfrak{U}_X = F^{-1}(\mathfrak{V}_X)$, then $\mathfrak{U}_X$ is an open nbd. of X in $C_G^r(TM)$.

Define $A_X \subset \mathfrak{F}_G^r(M)$ by:

$$A_X = \{\{k^t\} \varepsilon W_X : k^t|q = F_t^X|q, \ t \varepsilon R\}.$$

We may suppose that for all elements of A_X, q is 2*-

generic.

We have from P-H-S $\underline{1}$, that the map:

$$E: A_X \longrightarrow \mathfrak{C}^r_G(\mathbb{N}^s_\delta,\ M); \{f^t\} \longmapsto I^s_{loc}(\{f^t\}), \text{is continuous}$$

(This is strictly a parametrised P-H-S $\underline{1}$ over q). Here $I^s_{loc}(\{f^t\})$ gives the local parametrisation for the local stable manifold of G(q), w.r.t. $\{f^t\}$.

We may suppose that, with the notation of theorem 26, $P^q_s \subset \mathbb{N}^s_{\delta/4}$ is a fundamental domain for elements of A_X pulled back to $\mathbb{N}^s$, since $\mathbb{F}^X_t$ is transversal to P^q_s.

Thus the map:

$$H: A_X \longrightarrow \mathfrak{C}^r_G(\mathbb{N}^s, M); \{f^t\} \longmapsto I^s(\{f^t\}), \text{where } I^s \text{ gives}$$

global stable manifold, is continuous (i.e. we are here using a fixed fundamental domain in $\mathbb{N}^s$, together with a fixed compression of $\mathbb{N}^s$)

Now we note that the function $K: U_X \longrightarrow \mathfrak{Diff}^r_G(M)$, induces a map $\tilde{K}: V_X \longrightarrow \mathfrak{F}^r_G(M)$, defined by:

$$\tilde{K}(\{F^Z_t\}) = \{K(Z).F^Z_t.K(Z)^{-1}\} \stackrel{def}{=} \{\tilde{F}^Z_t\}.$$

It is easy to see that $\tilde{K}$ is continuous & so, for possibly smaller U_X, we may insist that $\tilde{K}(V_X) \subset A_X$, by the definition of K.

Now associated to $\{\tilde{F}^Z_t\}$ we have the parametrization map $I^s(\{\tilde{F}^Z_t\})$, & this parametrizises the stable manifold of q w.r.t. the flow $\{\tilde{F}^Z_t\}$, hence $K(Z)^{-1}.I^s(\{\tilde{F}^Z_t\})$ is a parametrization for the stable manifold of G(q(Z)) for Z. Since K depends continuously on Z we may state:

Theorem 27

Let q be a 2*-generic closed orbit for $X\in C^r_G(TM)$. If $\bar{N}^s$ denotes a C^r G vector bundle approximation to $\bar{N}^s$, then there exists a nbd. $N(X,q)$ of X in $C^r_G(TM)$ & a map I^s s.t.:

$$I^s : N(X,q) \longrightarrow \mathbb{C}^r_G(\bar{N}^s, M) \text{ is a continuous map \&}$$

$I^s(Z)$ is a parametrisation for the stable manifold of $G(q(Z))$.

Similarly for the unstable manifold & for singular 1*-generic sets.

Remark:

1. In the terminology of Abraham 1, I^s is a C^r pseudo-representation.

2. If $Y\in N(X,q)$, we may express our parametrisation $I^s(Y)$:

$$I^s(Y)e = F^Y_{T(e)}\cdot K(Y)^{-1}\cdot I^s_{loc}(\{K(Y)\cdot F^Y_t\cdot K(Y)^{-1}\})\cdot \bar{F}^{Y,K}_{T(e)}(e),\ e\in\bar{N}^s,$$

where $\bar{F}^{Y,K}$ is the pull back to $\bar{N}^s$ of F^Y, & depends on K.

Now recall that $I^s_{loc}(\{K(Y)\cdot F^Y_t\cdot K(Y)^{-1}\})$ is defined on $\bar{N}^s_\delta$, & we may assume that we have a G-invariant nbd. V_q of $G(q)$ s.t.:

$$K(Y)^{-1}\cdot I^s_{loc}(\{K(Y)\cdot F^Y_t\cdot K(Y)^{-1}\})(\bar{N}^s_\delta)\subset V_q,\quad Y\in N(X,q).$$

Here we note that we may assume that V_q is arbitrarily small, by choosing 'δ' small.

Fix $Z\in N(X,q)$. Then if we consider the set $A(Z)$ of $Y\in N(X,q)$, s.t. $Y=Z$ on V_q, we have for such Y:

1. $K(Y)=K(Z)$; 2. $F^{Y,K}=F^{Z,K}$; 3. $I^s_{loc}(\{K(Y)\cdot F^Y_t\cdot K(Y)^{-1}\})$ is independent of $Y\in A(Z)$; 4. 'T' is independent of $Y\in A(Z)$.

Thus we have for such Y:

$$I^s(Y)(e) = F^Y(G(Z,e)),$$ where $G(Z,e)$ is a C^r function of e & independent of Y.

We may suppose that our nbd. V_q is chosen both for the unstable & stable manifolds of $G(q)$ to satisfy the above.

3. In the sequel, we will usually write 'N^s' to denote the C^r approximation '$\bar{N}^s$'.

12. Generalised Equivariant Hartman's Theorems.

As pointed out in the previous section, most of this section is an easy application of P-H-S 1: We prove a Hartman's theorem for equivariant diffeomorphisms & then one for equivariant vectorfields, indicating what adaptations have to be made to P-H-S 1.

Hartman's Theorem for Equivariant Diffeomorphisms

The aim of this section is to prove:

Theorem 28

If V is a G,r normally hyperbolic set for $\mathfrak{Diff}^r_G(M)$, then f is conjugate to Nf, near V, by an equivariant homeomorphism.

Here $Nf = Tf|N^s \oplus N^u$.

Lemma 11

Let

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d] & E \arrow[d] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

be a G-Banach bundle automorphism, covering the equivariant homeomorphism f. Let $F': E \longrightarrow E$ be a C^0 equivariant map & be s.t.:

a) F' covers f.

b) $L(F'_x - F_x) < d$, $|F'_x - F_x| < d'$. $\forall x \in X$, for "d,d'", see P-H-S 1.

c) $F'(0_x) = 0_{fx}$, $x \in X$.

Where F_x, F'_x are F, F' restricted to E_x, also 0_x is the origin of E_x. Then F' is conjugate to F. The conjugacy leaves X fixed & preserves E-fibers. It is the unique conjugacy at a finite distance from the identity. Further it is equivariant.

Proof

Everything, except the last statement, is in P-H-S 1. The last statement follows since, if h is the conjugacy it may may be checked that $g^{-1}hg$ is also a conjugacy, $\forall g \in G$. Hence, by the uniqueness, h is equivariant.

Using Lemma 11, we may prove easily, as in P-H-S 1:

Lemma 12

Let F, F' be as in Lemma 11, except that F' is defined only on U, a uniform G-invariant nbd. of a closed G & f invariant subset $X_o \subset X$, & F' satisfies:

a) F' covers f equaling f on $X \cap U$.

b) $L(F'_x - F_x) < d/2, x \in X \cap U$.

The function $F'_x - F_x$ being defined only on $U \cap E_x$. Then the restrictions of F' & F to nbds of X_o are equivariantly conjugate. The conjugacy equals the identity on X & preserves fibers.

Next we recall from Palis & Smale 1, the idea of setting up a C^1 regular f-invariant fibration, transverse to the stable manifold (i.e. Local stable manifold) of a normally hyperbolic set V. The key ideas here are firstly to set up an f-invariant C^1 transverse fibration over a nbd. of a fundamental domain in the local stable manifold. Having done this we iterate by f & use the λ-lemma to extend the fibration regularly to V, with fiber, at $p \in V, W^{uu}_{loc}(p)$.

Here we wish to show that, given an equivariant fundamental domain of the local stable manifold of a G, r normally hyperbolic set V, we can construct an f & G invariant

C^1 fibration transverse to it, on a nbd. of the given fundamental domain in $W^s_{loc}(V)$.

To do this it is sufficient to prove a G-version of the C^r retraction Lemma (Palis-Smale 1).

Proposition(Smale)

If N is a C^r G-invariant submanifold (with boundary) $I=[0,1]$.

Then there exists a nbd. $U(\Delta)$ of $\Delta(N)\subset N \times N$ & a C^r equivariant function $\eta: U(\Delta) \times I \longrightarrow N$ s.t.:

1. $\eta(x,y,0)=x$.
2. $\eta(x,y,1)=y$.
3. $\eta(x,x,t)=x$, $t\in I$.
4. If $x,y\in\partial N, \eta(x,y,t)\in\partial N$.

Proof

Choose a C^∞ structure on N compatible with its C^r structure. Define η via geodesics in some C^∞ G-Riemannian metric, for which ∂N is totally geodesic.

Lemma (C^r G-retraction Lemma)

M: C^∞ G-manifold, B a G-invariant closed C^r submanifold.

$A\subset B$ as a G-invariant compact set, U_0 is a G-invariant nbd. of A in M, $r_0: U_0 \longrightarrow B$ a C^r G-retraction onto $U_0\cap B$, then there exists a G-invariant nbd. U of B & an equivariant retraction $r: U\longrightarrow B$, s.t. $r|U_0' = r_0|U_0'$, where U_0' is a G-invariant nbd. of A.

Further, if B has bndry, the above holds with $A\subset \text{int}(B)$, & U is a G-tubular nbd. with bndry ∂U, fibered by r over ∂B.

<u>Proof</u>

The tubular nbd. theorem yields a C^r equivariant retraction $p:U_1 \longrightarrow B$, where U_1 is a G-tubular nbd. of B. Let b be a C^∞ equivariant bump function, with $b=1$ on a nbd. of A, with $\mathrm{supp}(b) \subset U_0$ & $0 \leq b \leq 1$.

Set:

$$r(x) = p(x), \text{ if } x \notin U_0,$$
$$= \eta(p(x), r_0(x); b(x)), \text{ if } x \in U_0.$$

Using this result we may easily show, using the λ-lemma, that we have a C^1 regular G & f invariant fibration transverse to $W^s(V)$, which has the property that the fiber over $p \in V$ is $W^{uu}_{loc}(p)$.

<u>Proof of Theorem 28</u>

We proceed exactly as in P-H-S <u>1</u>, using the appropriate G-version of the lemmas used there, which have been proved above. We omit details.

<u>Hartman's Theorem for equivariant Vectorfields</u>

We wish to prove:

<u>Theorem 29</u>

If $\{f^t\}$ is a G,r normally hyperbolic flow at V, then $\{f^t\}$ is conjugate to $\{Nf^t\}$ by an equivariant homeomorphism h, which is independent of t.

There are two main ingredients in the proof of this theorem. Firstly 'C^1 equivariant fundamental domains for flows', which we have already contructed in the previous section; Secondly the following local Lemma of P-H-S 1, adapted to the equivariant case:

Lemma 13

Let $\{F^t\}$ be a continuous linear equivariant flow on a G-Banach bundle $p:E\longrightarrow X$, covering the flow $\{f^t\}$ on the base X. Suppose the time one map is a uniform contraction on fibers:

$$\|F^1|E_x\|\leq k<1,\ \forall x\in X.$$

Let U be a negatively invariant non-empty G-invariant subset of X. Let $\{G^t\}$ be a local equivariant flow over U, also covering $\{f^t\}$ leaving the zero section of E invariant & being close to $\{F^t\}$:

$$L(F^t-G^t|E_x)\leq\mu\leq\min(\omega/3,1-k),\ 0\leq t\leq 1,\ x\in X,\ \text{for}$$

$\omega=\inf\{m(F^t|E_x):x\in X,1>t>0\}$. Then $\{F^t\}$ & $\{G^t\}$ are conjugate by an equivariant conjugacy near the zero section, similarly for a uniform expahsion.

Proof

Just a question of checking P-H-S 1 to see that their proof works for the equivariant case. In particular we note that the functions:

$$g(y)=\int_0^1\|G^t(y)\|dt\ \&\ f(y)=\int_0^1\|F^ty\|dt$$

defined in P-H-S 1 are both equivariant.

Given the above lemma we may then proceed to the proof of theorem 29 exactly as in P-H-S 1. We omit details.

Now using the above equivariant versions of Hartman's theorem, we see that we have a complete description, up to topological conjugacy, of the flow of an equivariant vector field (or diffeomorphism) in a nbd. of a generic critical set. Thus if $G(x)$ is a 1-generic singular set, we essentially have '$G(x)$-worth' of ordinary generic singular flow behaviour in a nbd of a generic fixed point, i.e. it is sufficient to study $Nf_t|N_y$, for any $y \in G(x)$. For 2-generic closed orbits q, it is sufficient to study $Nf_t|(N|q)$.

There is left the problem of whether at the C^1 level any new features appear. No study is made of this point here, however, we note that if $G(x)$ is a generic singular set for $X \in C^r_G(TM)$, then to show that no new features appear at the C^s level, $0 \leq s \leq r$, it is sufficient to construct a C^s G tubular nbd. of $G(x)$ which is invariant by the flow (i.e. X is tangent to the fibers). This is what we have done above for $s=0$.

13. '2*-generic' is a generic property

The aim of this section is to prove the above proposition. Our proof will be similar to that in Peixoto 1, for the $G=\mathrm{id}$ case, though somewhat more complicated, involving a triple induction on period & orbit type.

We recall, from section 6, that we have a simple order on the orbit types of M. We denote this simple order by $\leqslant$. If $M=M_1\cup\ldots\cup M_m$, then we write $M^i=\bigcup_{j\leqslant i} M_j$. If $W\subset M$, then we give W_i & W^i the obvious meanings. Note that $M^m=M$, $W^m=W$.

Definition 32

For $T>0$, let

$\mathfrak{b}^i(T)=\{X\in C^r_G(TM): X$ is 1*-generic & all closed orbits of period $\leqslant T$, with type $\leqslant i$, are 2*-generic$\}$.

We set $\mathfrak{b}(T)=\mathfrak{b}^m(T)$.

We first state, & prove where necessary, analogues of three lemmas in Peixoto 1.

Lemma 14

If $G(x)$ is a 1*-generic singular set of $X\in\mathfrak{X}^1_G{}^*(M;r)$ & $T>0$, then there is a G-tubular nbd. U of $G(x)$ in M & a nbd. $\mathfrak{U}$ of X in $\mathfrak{X}^1_G{}^*(M;r)$, s.t. whenever $Y\in\mathfrak{U}$, then Y has in U exactly one singular set $G(x(Y))$, which depends continuously on Y, & every closed orbit of Y, meeting U has period $>T$.

Proof

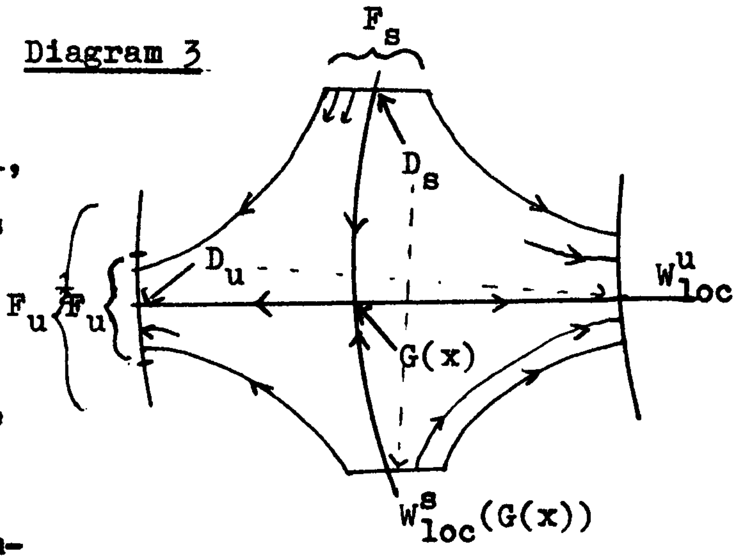
The first part, concerning U & $x(Y)$, follows from previous work: essentially Theorem 7.

To prove the second part, we present a rather extended argument, which will generalise immediately to prove part of Lemma 15. The main ingredient here is generalised Hartman's Theorem.

First we note that $\{F_t^X\}$ is conjugate to $\{NF_t^X\}$ in some invariant nbd. J of G(x). Working inside J we choose C^1-equivariant fundamental domains D_s & D_u for $\{F_t^X\}$ in $W^s_{loc}(G(x))$ & $W^u_{loc}(G(x))$ respectively.

We take transverse G-invariant fibrations F_s & F_u to D_s & D_u respectively as in section 11. We will assume that F_s & F_u are closed & that they are transverse to the flow of X-see Diagram 3.

Diagram 3



We suppose F_s is chosen sufficiently small, s.t. the flow of X gives us a map $\Phi(X)$ from the boundary of F_s to $\frac{1}{2}F_u$, where by $\frac{1}{2}F_u$, we mean the fibration defined by F_u, with fiber half the diameter of F_u. Then $\Phi(\text{bndry}(F_s))$ divides F_u into two components one meeting $W^u_{loc}(G(x))$.

We denote this component by F_u^1, & set S(X)=union of F_u^1, F_s & the set swept out by bndry(F_s). Now also F_s is swept out by the flow of X into a nbd. of $W^u_{loc}(G(x))$, which we will denote by N(X), also we set $N^+(X)=N(X)\cup W^u_{loc}(G(X))$.

Now if $z\varepsilon F_s$, the positive orbit through z either tends to G(x) or, for some smallest $T_X(z)\varepsilon R^+$, $F^X_{T_X(z)}(z)\varepsilon F_u^1$, & the orbit of z over the interval $(0,T_X(z))$ is contained in $N^+(X)$. Thus we may define $T_X:F_s \longrightarrow R^+\cup\{\infty\}$, in the obvious way.

Now theorem 29 allows us to choose F_s sufficiently small such that $T_X(z)>2T, \forall z\varepsilon F_s$.

Since F_s is transversal to the flow of X, we may, w.l.o.g. suppose that we have a nbd. $\mathfrak{U}$ of X, s.t. if $Y\varepsilon\mathfrak{U}$, Y is transversal to F_s & F_u & further $\Phi(Y)$:bndry$(F_s)\longrightarrow F_u$ is still defined. By theorem 27 we may assume $W^s_{loc}(G(x(Y)))$ meets F_s close to D_s.

Thus, for possibly smaller $\mathfrak{U}$:

1. $^+(Y)$ is still a nbd. of $G(x(Y))$ &, by considering $S(Y)$, we may insist that $N^+(Y)\supset U$, where U is some G-tubular nbd. of $G(x)$.

2. We may also insist that $T_Y(z)$, for $Y\varepsilon\mathfrak{U}$, is $>T$.

1. & 2. together imply that if $Y\varepsilon\mathfrak{U}$, then there is no closed orbit of Y meeting U with period $\leq T$.

Lemma 15

Let $T>0$ & q be a 2*-generic closed orbit of $X\varepsilon\mathfrak{X}^1_G{}^*(M;r)$ with period $<T$. Then there is a G-tubular nbd. V of G(q) & a nbd. $\mathfrak{V}$ of X in $\mathfrak{X}^1_G{}^*(M;r)$ s.t. every $Y\varepsilon\mathfrak{V}$ has a 2*-generic closed orbit $q(Y)\subset V$ & besides, with the eventual exception of $G(q(Y))$, every closed orbit of Y meeting V has period $\geq T$. $G(q(Y))$ varies continuously with Y.

Proof

If q is an orbit of type A, then f_X, the generalised Poincare map for q is 1*-generic on $G(x)$, $x\varepsilon q$, & the proof follows formally as in Peixoto 1.

Suppose q is an orbit of type B. Then using the work preceeding proposition 28, we certainly may find a nbd. $\mathfrak{U}_1$ of X & a G-invariant nbd. V_1 of G(q) such that we have a 2*-generic closed orbit q(Y) of type B, depending continuously on $Y \in \mathfrak{U}_1$ & lying in $V \cap M_i$, where type(q)=i.

Now using generalised Hartman's theorem, we have an invariant nbd. J of G(q) s.t. $\{F_t^X\}$ is conjugate to $\{NF_t^X\}$ on J. We may then repeat the argument of Lemma 14, using suitable transverse fibrations to fundamental domains of $W^s_{loc}(G(q))$ & $W^u_{loc}(G(q))$ to obtain the required result for suitable V & $\mathfrak{U}$.

Remark

Essentially the problem here is to show that no orbits of type A appear near q, under perturbation of X. We might as well have proceeded by noting that if, with the above notation, diam(J)=d (fiberwise) then we could have found a nbd. $\mathfrak{W}$ of X s.t. if $Y \in \mathfrak{W}$, we could suppose that the diameter of the corresponding conjugating nbd. of G(q(Y)) was $\geqslant d/2$-this follows by examination of the proof of Hartman's theorem-& then proceeded in a similar manner to which we treated type A closed orbits.

As an immediate corollary of Lemma 15 we have:

Corollary 15.1

1. If $X \in \mathfrak{b}(T)$, then X has only a finite number of closed orbits (mod G) of period $\leqslant T$.

2. If $X \in \mathfrak{b}^i(T)$, then X has only a finite number of closed orbits in M^i of period $\leqslant T$, mod G of course!

Lemma 16

Let K be a compact G-invariant subset of M & assume that no point of K is a singular point or belongs to a closed orbit, of period $\leq T$, of a vectorfield $X \in C^r_G(TM)$. Then:

1. $\exists$ a nbd. $\mathfrak{W}$ of X in $C^r(TM)$, s.t. every closed orbit of $Y \in \mathfrak{W}$, meeting K, has period $> T$.

2. In fact we may assume that we have an invariant nbd. W of K & nbd. $\mathfrak{W}$ of X, s.t. every closed orbit of $Y \in \mathfrak{W}$, meeting W, has period $> T$.

Proof

1. is just Lemma 3 of Peixoto 1, & 2. is a consequence of the proof.

We now prove:

Proposition 33.

$\mathfrak{b}^i(T)$ is open in $C^r_G(TM)$.

Proof

Let $p \in M^i$ be s.t. it is neither a singularity of $X \in \mathfrak{b}^i(T)$ or is situated on a closed orbit (lying in M^i) of period $\leq T$. Let $\mathfrak{W}$ be a nbd. of X in $\mathfrak{N}^1_G{}^*(M;r)$ & W be a nbd. of p be s.t. whenever $Y \in \mathfrak{W}$, every closed orbit of Y meeting W^i has period $> T$. If p is a singularity of X or is a closed orbit of period $\leq T$, lying in M^i, we apply Lemma 14 or Lemma 15 to get nbds $U, \mathfrak{U}$ or $V, \mathfrak{V}$.

Since M^i is compact we find a finite number of U, V & W covering M^i.

The intersection of $\mathfrak{U}, \mathfrak{V}$ & $\mathfrak{W}$ contains an open set made up of points of $\mathfrak{b}^i(T)$.

Definition 33

Let $\mathfrak{g}^i(T)=\{X\varepsilon\mathfrak{b}^{i-1}(T)$:all closed orbits of X of type B, lying in M_i, of period $\leq T$, are 2*-generic$\}$.

Here we set $\mathfrak{b}^o(T)=\mathfrak{R}_G^{1*}(M;r)$ & we have:

$$\mathfrak{b}^o(T)\subset\mathfrak{g}^1(T)\subset\mathfrak{b}^1(T)\subset\ldots\subset\mathfrak{g}^m(T)\subset\mathfrak{b}^m(T)=\mathfrak{b}(T).$$

Lemma 17

Let K be a compact invariant subset of M & let $X\varepsilon C_G^r(TM)$. Suppose that the X-invariant G-orbits in K^i are filled with X orbits of period $>T$, where $T>0$.

Then we assert that there exists a nbd. $\mathfrak{p}$ of X & an invariant nbd. P of K^i, s.t. if $Y\varepsilon\mathfrak{p}$, then there are no closed orbits of Y, of type B, lying in P, of period $\leq T$.

Proof

Let I denote the set of X-invariant G-orbits in K^i. I is closed & therefore compact (look at the orbit space). We apply Lemma 16, part 2., to I, to get an open invariant nbd. P' of I & a nbd. $\mathfrak{p}'$ of X.

Suppose $y\varepsilon K^i-I$, then we assert that we may find a nbd. $\mathfrak{p}_y$ of X & an invariant nbd. P_y of y, s.t. if $Y\varepsilon\mathfrak{p}_y$, no invariant G-orbits, lying in M^i, meet P_y:

Now we certainly have a closed G-tubular nbd. $\mathfrak{P}_y$ of y, which satisfies the required property for X. In addition, since y is not a singular point, for sufficiently small $\mathfrak{P}_y$, $\exists e>0$, s.t. $\forall z\varepsilon\mathfrak{P}_y$, G(z) is not invariant by F_e^X.

Suppose there is no nbd. $\mathfrak{p}_y$ of X, for which $\mathfrak{P}_y$ 'works'. Then we may find a sequence of vectorfields $Y_K\longrightarrow X$, s.t.

each Y_k has at least one invariant set $G(x_k)$ in $\bar{P}_y$. Since $\bar{P}_y$ is compact we will assume, w.l.o.g., that x_k converges to $z\varepsilon\bar{P}_y$. Now we note that $F_e^k = F_e^{Y_k}$ converges to F_e^X, in the C^o sense.

x_k defines a convergent sequence $\bar{x}_k$ converging to $\bar{z}$, in the compact space $\bar{P}_y/G$.

We have $\bar{F}_e^k(\bar{x}_k)=\bar{x}_k$, where $\bar{F}^k$ is the flow induced on the orbit space.

Now, by the definition of $\bar{P}_y, \bar{F}_e^X(z)\neq z$. But $\bar{F}_e^k \longrightarrow \bar{F}_e^X$, contradiction. Therefore, using the 1st Countability of $C_G^r(TM)$, we may find nbds. $\mathfrak{P}_y$ & P_y as required.

Thus we have a cover of K^i by P' & $\{P_y\}_{y\varepsilon K^i-P'}$, & hence a finite subcover, since K^i is compact. We then take the finite intersection of the resulting '$\mathfrak{P}$' & the union of the corresponding 'P' to give the required result.

__Proposition 34__

$\mathfrak{b}(T)$ is dense in $\mathfrak{R}_G^{1*}(M;r)$.

__Proof__

We will proceed by induction on i, proving:

1. $\mathfrak{s}^1(T)\subset\mathfrak{b}^o(T)$, as an open dense set.
2. $\mathfrak{b}^1(T)\subset\mathfrak{s}^1(T)$, " " " " " "
3. $\mathfrak{s}^i(T)\subset\mathfrak{b}^{i-1}(T)$, " " " " " ", $1\leqslant i\leqslant m$.
4. $\mathfrak{b}^i(T)\subset\mathfrak{s}^i(T)$, " " " " " ", $1\leqslant i\leqslant m-1$.

1,2,3 & 4 together prove $\mathfrak{b}(T)$ is dense.

1. __$\mathfrak{s}^1(T)\subset\mathfrak{b}^o(T)$ as an open dense set.__

A. __Openness__

Suppose $X\varepsilon\mathfrak{s}^1(T)$. Let $z\varepsilon M_1$ be s.t. z does not lie on an

X-invariant orbit. Lemma 17 gives us nbds $\mathfrak{p}$ of X & P of G(z).

Next we note that the subset of M_1 consisting of X-invariant G-orbits is closed. Either $\mathrm{rank}(N(G_x)/G_x)=1$ or not, $x\in M_1$.

If $\mathrm{rank}(N(G_x)/G_x)=1$, then our X-invariant sets are all filled with 2*-generic closed orbits & are isolated. We may apply Lemma 15 to get nbds V & $\mathfrak{v}$.

If $\mathrm{rank}(N(G_x)/G_x)>1$, then our invariant sets are all filled with orbits of period $>T$ & we may apply Lemma 16, part 2., to get nbds W & $\mathfrak{w}$.

If $\mathrm{rank}(N(G_x)/G_x)=0$, then our X-invariant sets are all 1*-generic singular sets & we may apply Lemma 14 to get nbds U & $\mathfrak{u}$.

Since M^1 is compact we have a finite subcover by nbds P & U or V or W, taking the corresponding intersection of $\mathfrak{p}$ & $\mathfrak{v}$ or $\mathfrak{w}$ or $\mathfrak{u}$ we get that $\mathfrak{g}^1(T)$ is open.

B. <u>Density</u>

Let $X\in\mathfrak{b}^0(T)$.

First let $d_1>0$, be such that no closed orbit of X has period less than d_1. Such d_1 exist, for otherwise we could find a sequence of closed orbits of X, whose periods tend to zero, converging to a singular set of X. By generalised Hartman's theorem this is impossible.

Let $I_1(T)$ denote the set of closed orbits of X, of type B, with periods $\leq T$, lying in M_1. Then $I_1(T)$ is a closed subset of M_1, disjoint from the singular set of X: S(X) (in M_1).

Using Proposition 30, we assign to each orbit of $I_1(T)$ a G-tubular nbd. pair (U,V) & Poincare disc pair (D_1,D_2).

We suppose U is chosen sufficiently small so that it meets each orbit of $I_1(T)$ at most once & is disjoint from a nbd. of $S(X)$.

Using the compactness of $I_1(T)$, we extract a finite subcover of $I_1(T)$, $\{V^i\}_{i=1,..,k}$.

Using Proposition 30, we may make a C^r small change in X, vanishing outside U^1, s.t. all closed orbits of type B, lying in $\bar{V}^1 \cap M_1$ are 2*-generic. We call the perturbed vector field Z_1.

Again, using Proposition 30, we make a C^r small change in Z_1, to Z_2, vanishing outside U^2, s.t. all closed orbits of type B, lying in $\bar{V}^2 \cap M_1$ are 2*-generic. From Lemmas 15 & 17 we do not disturb the 2* generic situation we had before in $\bar{V}^1 \cap M_1$.

Repeating this proceedure, we get Z_k, s.t. Z_k is C^r close to X & all closed orbits of type B, lying in $M_1 \cap (\bigcup V^i)$ are 2*-generic.

We assert that we may assume that no new orbits of type B, period $\leq$ T, appear in $M_1 - \bigcup V^i$: But this follows from Lemma 17.

Thus $\mathfrak{S}^1(T)$ is dense.

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2. $\mathfrak{b}^1(T) \subset \mathfrak{S}^1(T)$ as an open dense set.

We have already proved openness: We have therefore only to prove density.

Let $X \in \mathfrak{S}^1(T)$ & let $d_1 > 0$ be a lower bound for the periods of the closed orbits of X.

Let $\Gamma^1 = \Gamma^1(d_1, 3d_1/2)$ be the set of all closed orbits of X, lying in M_1, whose periods are within the closed interval $[d_1, 3d_1/2]$. Clearly Γ^1 is a closed set in M & $\Gamma^1 = \Gamma^1_A \cup \Gamma^1_B$, where Γ^1_A consists of closed orbits of type A & Γ^1_B consists of closed orbits of type B. Since Γ^1_B consists of a finite (mod G) number of 2*-generic closed orbits we have (Lemma 15) an invariant nbd. $N_1(B)$ of Γ^1_B, disjoint from an invariant nbd. of Γ^1_A, consequently Γ^1_A is closed & so compact.

We assign to every orbit family $G(q)$ Γ^1_A a G-tubular nbd. triple (U,V,W) of $G(q)$ & a corresponding Poincare section triple (D_1, D_2, D_3), chosen as in Proposition 32. We choose U so small that D_1 meets trajectories of Γ^1 at most once & the time $T(y)$ along the X-orbit from D_2 to D_1, $y \in D_2$, satisfies $(7/8)d_1 \leq T(y) < (13/8)d_1$. Further, we may suppose w.l.o.g., that U does not meet $N_1(B)$.

Since Γ^1_A is compact we can extract from the covering a finite set $(U^i, V^i, W^i)_{i=1,\dots,k}$, s.t. the W^i's cover Γ^1_A. Then, using Proposition 32, we may make a C^r-small change in X, vanishing outside U^1, to get a vector field Y_1^1, C^r close to X, such that every closed orbit of Y_1^1, of period $\leq 3d_1/2$, lying in M_1, & meeting $\overline{W}^1$ is 2*-generic, also we have that the Poincare map of $Y_1^1 | (D_2 \cap M_1)$ is generic.

An important point here is that, by taking Y_1^1 sufficiently close to X, every closed orbit of Y_1^1 meeting $\overline{W}^1 \cap M_1$, which corresponds to m-turns, $m > 1$, has period close to $m\bar{d}$, where $(7/8)d_1 \leq \bar{d} < (13/8)d_1$, & so greater than $3d_1/2$. Thus every trajectory, corresponding to $m=1$, meeting $\bar{W}^1 \cap M_1$ is 2^*-generic.

We now proceed as above & perturb Y_1' slightly inside U^2, getting a vector field Y_2', such that all trajectories of Y_2' meeting $\bar{W}^2 \cap M_1$, of period $\leqslant 3d_1/2$ are 2^*-generic. Lemmas 15 & 16 allow us to perturb Y_1' to Y_2' & not disturb the generic situation we had before in $\bar{W}^1$.

Repeating this argument up to V^k, we obtain $Y_1 = Y_k'$, such that Y_1 is arbitrarily close to X, & besides all of its trajectories of period $\leqslant 3d_1/2$, contained in $W_1' = \bigcup W^i$ & lying in M_1 are 2^*-generic. Outside W_1', Y_1 might have nongeneric closed orbits of period $\leqslant 3d_1/2$, lying in M_1, but applying Lemma 16, with $K = (M - W_1' - N_1(B))_1$, we see that for Y_1 close enough to X, all periodic orbits of Y_1 meeting K have period $> 3d_1/2$. So all periodic orbits of Y_1, contained in M_1, of period $\leqslant 3d_1/2$ are 2^*-generic.

We now essentially repeat this proceedure & work with the set $\Gamma_A^1(3d_1/2, 2d_1)$ of all closed orbits of Y_1, of type A, lying in M_1, within the interval $[3d_1/2, 2d_1]$. Compared with the previous case, there is a difference that Y_1 may have periodic orbits of period $< 3d_1/2$, lying in M_1, whereas X had none of period $< d_1$, but these orbits are generic & finite in number. Thus we can find a nbd. W of their union disjoint from the U's employed in covering $\Gamma_A^1(3d_1/2, 2d_1)$. We then do as before taking $K = (M - W_1' - W - N_1(B))_1$ & get Y_2 s.t. all of its closed orbits of period $\leqslant 2d_1$, lying in M_1, are 2^*-generic.

Proceeding this way we have $l \in Z$, s.t. $ld_1/2 > T$. $Y_l = Y$ can be made arbitrarily close to X & s.t. all of its closed orbits of period T are 2^*-generic (i.e. those that lie in M_1).i.e. $Y\in\mathfrak{U}^1(T)$.This proves 2.

3. <u>$\mathfrak{G}^i(T)\subset\mathfrak{U}^{i-1}(T)$ as an open dense set $1\leqslant i\leqslant m$</u>

We have the following sublemma:

<u>Sublemma</u>

Let $X\in\mathfrak{U}^j(T)$, $j\geqslant 1$, then there exists an open nbd. P of M^j in M & a nbd. $\mathfrak{P}$ of X in $\mathfrak{U}^j(T)$, s.t. if $Z\in\mathfrak{P}$, all closed orbits of Z, meeting P of period $\leqslant T$, lie in M^j & are 2^*-generic.

<u>Proof</u>

To prove this all we have to show is that we may find P' & $\mathfrak{P}'$ s,t. for $Z\in\mathfrak{P}$, Z has no closed orbits of period $\leqslant T$, meeting $(M-M^j)\cap P$, the result will then follow easily from the openness of $\mathfrak{U}^j(T)$, we assume we have proved openness for $\mathfrak{U}^i$, $i\leqslant j$.

But this is easy, using Lemmas 14, 15 & 16; we omit details.

Having this sublemma, we then look at $X\in\mathfrak{U}^{i-1}(T)$ & consider M_i-P, where P is calculated for $i-1$. M_i-P is closed & therefore compact. We may then repeat the argument we gave for $i=1$, restricted to M_i-P & $\mathfrak{P}$, we omit details.

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4. <u>$\mathfrak{U}^i(T)\subset\mathfrak{G}^i(T)$ as an open & dense set, $1\leqslant i\leqslant m-1$.</u>

We note that type B orbits in M_i, period $\leqslant T$, are disjoint from type A orbits, period $\leqslant T$ & both sets are closed. Using the sublemma & this remark we consider M_i-P & repeat the argument of 2. to prove 4.

Thus we have shown that $\mathfrak{U}(T)$ is an open & dense subset of $\mathcal{C}^r_G(TM)$, but:

$$\mathfrak{R}_G^{2^*}(M;r)=\bigcap_{T=1}^{\infty}\mathfrak{b}(T)$$ (T runs over the positive integers)

So therefore we have our main theorem:

Theorem 30

"2*-generic" is a generic property, i.e. $\mathfrak{R}_G^{2^*}(M;r)$ is a residual suset of $C_G^r(TM)$.

As an easy consequence of this result we have the corresponding result for diffeomorphisms:

Theorem 31

"2*-generic" for equivariant diffeomorphisms is a a generic property, i.e. $\mathfrak{A}_G^{2^*}(M;r)$ is a residual subset of $\mathfrak{Diff}_G^r(M)$.

14. Transversality for Invariant Manifolds in G-manifolds

In this section we will be concerned with defining a Transversality condition for the stable & unstable manifolds of generic critical elements of an equivariant vector field. We present reasons for our definitions, which justify their choice in this context.

If $W \subset M$ as a C^r G-invariant submanifold & $H \subset G$ as a closed subgroup, we define:

$$W_H = \{ w \in W : G_w = H \}.$$

Then W_H is a C^r submanifold of M. We note that even if W is closed, W_H is, in general, NOT closed.

Suppose W & V are C^r invariant submanifolds of M & let $y \in W \cap V$.

Definition 34

We say V meets W 'G transversally' at y iff:

$$W_{G_y} \mathrel{\bar{\pitchfork}_y} V_{G_y} \subset M_{G_y},$$

i.e. V & W meet transversally in M_{G_y}.

We write this '$W \bar{\pitchfork}^G_y V$'.

If all points of intersection of V & W are G transversal, we say W & V are G transversal & we write this, $W \bar{\pitchfork}^G V$.

Remark: It is not difficult to see that an equivalent formulation to the above is:

Definition 34'

$W \bar{\pitchfork}^G_y V$ iff $W_i \mathrel{\bar{\pitchfork}_y} V_i \subset M_i$, where type(y)=i.

Let J be a C^r equivariant injection of some G-manifold N into M & suppose that $W \subset M$ as a C^r invariant submanifold.

Definition 35

For $z \in N$, we say J is 'G transversal to W at z', iff $J: N_{G_z} \longrightarrow M_{G_z}$, is transversal to W_{G_z} at z; we write this:

$$J \pitchfork^G_z W.$$

If J is G transversal to W, $\forall z \in N$, we say that J is G transversal to W, & write this $J \pitchfork^G W$.

Remark:

1. Again $J \pitchfork^G_z W$ is equivalent to: $J: N_i \longrightarrow M_i$ is transversal to W_i at z, where type$(z)=i$.

2. We need that J is injective so that $G_z = G_{Jz}$; it is not sufficient for J to be an immersion, i.e. locally injective.

Suppose that p & q are generic critical elements of $X \in C^r_G(TM)$, i.e. either fixed points or closed orbits. We have C^r equivariant injective immersions:

$$I^u_p : N^u_p \longrightarrow M, \text{parametrising } W^u(G(p)).$$
$$I^s_q : N^s_q \longrightarrow M, \text{parametrising } W^s(G(q)).$$

Here N^u_p & N^s_q are the G-vector bundles defined in Theorem 26.

Thus we have a map:

$$I_{p,q} = I^u_p \times I^s_q : N^u_p \times N^s_q \longrightarrow M \times M.$$

Suppose $(y,y) \in \Delta(M)$ & type$(y)=i$.

Let $I^i_{p,q} = I_{p,q} | (N^u_p \times N^s_q)_i$, taking values in $(M \times M)_i$.

& $\hat{I}^{i}_{p,q}=I_{p,q}|(N^{u}_{p})_{i} \times (N^{s}_{q})_{i}$, taking values in $M_i \times M_i$.

Definition 36

If $I_{p,q}(z,v)=(y,y)$, we say that 'the unstable manifold of G(p) meets the stable manifold of G(q) at $y \in G$ transversally' iff:

$$I_{p,q} \pitchfork^{G}_{(z,v)} \Delta(M), \text{ \& we write this:}$$

$$W^{u}(G(p)) \pitchfork^{G}_{y} W^{s}(G(q)).$$

If all points of intersection of $W^u(G(p))$ & $W^s(G(q))$ are G transversal in this way, we write $W^u(G(p)) \pitchfork^G W^s(G(q))$.

Now, by definition, $I_{p,q} \pitchfork^{G}_{(z,v)} \Delta(M)$ iff:

$$I^{i}_{p,q} \pitchfork_{(z,v)} \Delta(M)_{i} \subset (M \times M)_{i} \quad \dots\dots\dots\dots \text{A}$$

We assert A is equivalent to:

$$\hat{I}^{i}_{p,q} \pitchfork_{(z,v)} \Delta(M_i) \subset M_i \times M_i \quad \dots\dots\dots\dots \text{B}$$

This follows by noting that:

1. $\Delta(M)_i = \Delta(M_i)$-trivial.
2. If C is a connected component of $\Delta(M_i)$, then the codim of C in $(M \times M)_i$ = codim of C in $M_i \times M_i$. That this is so follows since $M_i \times M_i$ is isolated, as a set of type i, in $(M \times M)_i$-see section 6.

To end this section we now give some examples which illustrate the definition & provide some justification for it.

Example 1

Let x be a 1-generic singular point for $X \in C^{r}_{G}(TM)$, then $W^u(G(x)) \pitchfork^{G}_{gx} W^s(G(x))$, $g \in G$; in fact these manifolds are actually transversal at $G(x)$.

To see the G transversality, just note that $X|M_i \in C^r_G(TM_i)$, where type$(x)=i$, is 1-generic at $G(x)$.

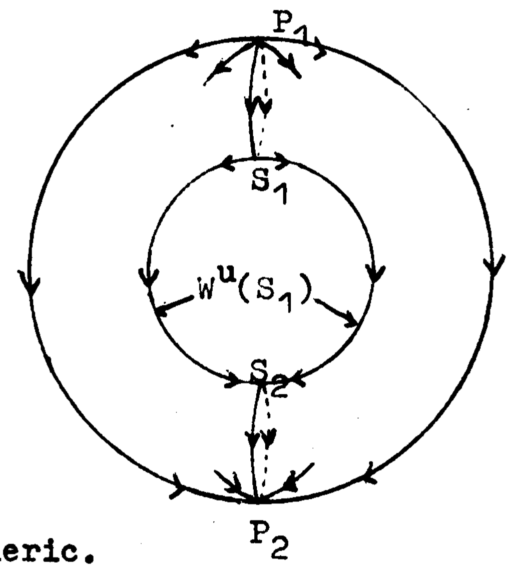
Similarly for 1-generic fixed sets of equivariant diffeomorphisms.

Example 2

Let q be a 2-generic closed orbit of X, then $W^u(G(q)) \pitchfork^G_y W^s(G(q))$, for $y \in q$; in fact the two manifolds are actually transversal at $G(q)$. This follows as above.

Example 3

We now give a specific example on the 2-torus. Essentially we have a $Z_2 \times Z_2$ action on T^2 & an equivariant gradient flow on T^2 defined by the height function:



One Z_2 is reflection in the plane of the paper, the other is normal to the paper in the plane defined by the normal & P_1, S_1, S_2 & P_2.

P_1 is a source, S_1 & S_2 are saddles & P_2 is an attractor. All are 1* generic.

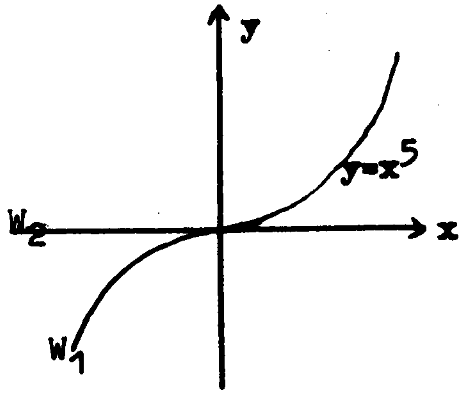
We note in particular:

1. The saddles S_1 & S_2 are joined by the unstable manifold of S_1. This would normally, of course, be consider as a non-generic situation, but here, it may be checked, they are $Z_2 \times Z_2$ transversal as are all the other stable & unstable manifolds.

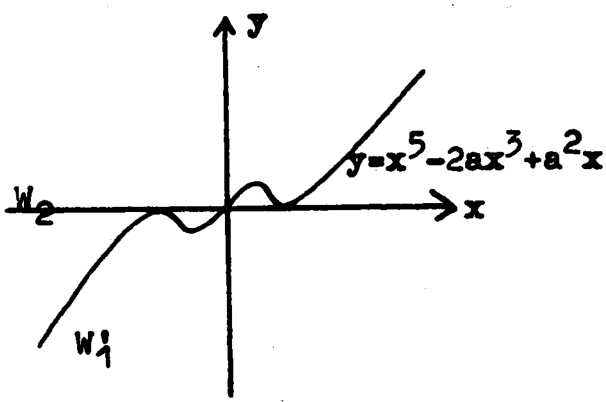
2. It is not difficult to show that the vectorfield is structurally stable-in the set of equivariant vector fields-in particular, under perturbation P_1, P_2, S_1, S_2 & the stable & unstable manifolds remain fixed.

Example 4

We take the Z_2 antipodal action in the plane & the two Z_2-invariant submanifolds W_1 & W_2 defined by $y=x^5$ & $y=0$, respectively:



Now W_1 & W_2 are Z_2 transversal. Define W_1' by $y=x^5+2bx^3+a^2x$, a & b small. Then W_1' is Z_2-invariant. Further, if $b<0$ & $b=-a$, then one may easily check that W_1' is not Z_2 transversal to W_1, see diagram, but since a & b may be chosen arbitrarily small, this shows that the Z_2-transversality condition is not open. One may easily construct similar examples for connected Lie groups, e.g. S^1.



Remark:

One might argue that by perturbing $y=x^5$ to $y=x^5+ax$, a small & positive, one would get openness of transversal intersection with W_2. However by considering the corresponding picture in R^3, with W_1 defined by $z=x^5+yx$, W_2 by $z=0$ & the Z_2

action defined to be the antipodal action in the (z,x) plane, y fixed.

Thus not only does example 4 show that we do not have the openness property of transversal intersection for G transversality but that the topological type of the intersection varies when we perturb a G-transversal map. This fact will prove to be embarassing & is primarily related to the fact that in the G-orbit type decomposition of a manifold some of the sets defined are non-compact.

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15. <u>'Generalised Kupka-Smale density theorem'</u>

In this section we define 3-genericity for equivariant vector fields & state & prove a general density theorem, generalising the Kupka-Smale density theorem.

<u>Definition 37</u>

Let $X\in\mathfrak{X}_G^{\mathfrak{o}*}(M;r)$. We say X is 3-generic iff the stable & unstable manifolds of the critical sets of X are G transversal.

We denote the set of such vector fields by $\mathfrak{X}_G^{3*}(M;r)$.

Similarly for equivariant diffeomorphisms. We denote the set of 3-generic diffeomorphisms by $\mathfrak{D}_G^{3*}(M;r)$.

The main objective of this section is to prove:

<u>Theorem 32</u>

$\mathfrak{X}_G^{3*}(M;r)\subset C_G^r(TM)$ as a residual set.

Our proof of this result will be similar to that given in Abraham 1. We give the proof in full as there are a number of new details.

We divide our proof into four parts:

<u>PART 1</u>

<u>Proposition A</u>

Let $X\in\mathfrak{b}(T)$, then there exists:

a) $\varepsilon>0$.

b) a nbd. N of X in $C_G^r(TM)$.

c) Open G-invariant subsets $V_1,\dots,V_k$ of M, with disjoint closures.

d) C^r pseudo representations $N\longrightarrow C^r(N_1^\sigma,M)$; $\sigma=s$ or u,

$l=1,..,k$;given by $Y \longrightarrow I_Y^{\sigma,l}$, s.t.:

1. $N \subset \mathfrak{b}(T+\varepsilon)$
2. For $Y \in N$, Y has exactly k critical elements, $p_Y^1,..,p_Y^k$ (mod G) of period $\leq T+\varepsilon$.(necessarily generic).
3. $G(p_Y^l) \subset V_l$, $l=1,..,k$; $Y \in N$.
4. $I_Y^{s,l}$ (resp. $I_Y^{u,l}$) is a parametrisation of the stable (resp. unstable) manifold of Y through $G(p_Y^l)$ $(l=1,..,k; Y \in N)$.
5. If $Y \in N$ & $x \notin \{G(p_Y^l)\}_{l=1,..,k}$, then $\exists t \in R$ s.t. $F_t^Y(x) \notin \bigcup V_l$

Proof

This is an easy combination of Proposition 33, Theorem 27, Lemma 14, Lemma 15 & Lemma 16.

PART 2

Fix $T \in R^+$. We recall, Proposition 33, that $\mathfrak{b}(T)$ is open in $C_G^r(TM)$, a separable Banach space. Thus there is a countable covering of $\mathfrak{b}(T)$ by open sets N_c, satisfying Proposition A, say:

$$\mathfrak{b}(T) = \bigcup_{c \in Z^+} N_c .$$

Since $C_G^r(TM)$ is paracompact (i.e. $C^r(TM)$ is metric) so is $\mathfrak{b}(T)$ & we may suppose that the cover $\{N_c\}_{c \in Z^+}$ is locally finite.

Thus for each $c \in Z^+$ we have defined:

a) A positive real number ε_c.
b) An open subset $N_c \subset \mathfrak{b}(T+\varepsilon_c) \subset \mathfrak{b}(T)$.
c) Open invariant subsets $V_c^1,..,V_c^{k_c}$ of M.
d) Generic critical elements $p_Y^1,...,p_Y^{k_c}$ of each $Y \in N_c$.

e) C^r pseudo representations $N_c \longrightarrow C^r(N_c^{\sigma,1},M)$ given by $Y \longmapsto I_{c,Y}^{\sigma,1}$, for $i=1,\dots,k_c$, $Y\in N_c$ & $\sigma=s$ or u.

Now since each $N_c^{\sigma,1}$ is a G vector bundle, we may define $N_c^{\sigma,1}(b)=\{v\in N_c^{\sigma,1}:\|v\|\leqslant b\ ,b\in R^+\}$.

We note that:

$$N_c^{\sigma,1}=\bigcup_{b=1}^{\infty} N_c^{\sigma,1}(b) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad 1.$$

& that $N_c^{\sigma,1}(b)$ is compact.

Let $W\subset N$ be a C^r G invariant submanifold of a G manifold N, where W is not necessarily closed. Suppose $n\in Z^+$ & we have an equivariant metric d on N.

We set:

$$W_i(n)=W_i-\{y\in N:d(y,\mathrm{bndry}(W_i))<1/n\}, \mathrm{bndry}(W_i)=\overline{W}_i-W_i.$$

Now W_i is, in general, not closed, but $W_i(n)$ is closed &, furthermore:

$W_i=\bigcup_{n\in Z^+}W_i(n)$, -here we use the fact that W_i is a submanifold of N.

Thus, for $n\in Z^+$, we may define:

$(N_c^{\sigma,1}(n))_i(n)$, which we will denote by $N_c^{\sigma,1}(n)_i$. Then $N_c^{\sigma,1}(n)_i$ is a compact set & we have:

$$(N_c^{\sigma,1})_i=\bigcup_{n\in Z^+}N_c^{\sigma,1}(n)_i$$

We will suppose for the moment that i is fixed.

Now for $c\in Z^+$, & $l,m=1,\dots,k_c$, let

$\mathfrak{M}(T,n,c,i;l,m)$ be the set of all $Y\in N_c$ s.t. :

$I_{1,m}\mathfrak{M}^G(M)$ on $N_c^{s,1}(n)_i \times N_c^{u,m}(n)_i$.

i.e., restricting attention to M_i, we require that $W^s(G(p_Y^1))$ be G-transversal to $W^u(G(p_Y^m))$ at least up to radius n of these manifolds, except, possibly within 1/n of their 'i-boundary'. Thus in M they will be G-transversal in M_i, except near the boundary of M_i in M.

We set:

$$\mathfrak{M}(T,n,i) = \bigcup_{c\in Z^+} \bigcap_{l,m=1}^{k_c} \mathfrak{M}(T,n,c,i;l,m)$$

Define:

$\mathfrak{M}(T,i) = \{X : X \in \mathfrak{b}(T)$ & all stable & unstable manifolds of critical elements of X, period $\leq$ T, are G transversal in $M_i\}$

&

$\mathfrak{M}(T) = \{X : X \in \mathfrak{b}(T)$ & all stable & unstable manifolds of critical elements, period $\leq$ T, are G-transversal in $M\}$.

Proposition B

$$\mathfrak{M}(T,i) = \bigcap_{n\in Z^+} \mathfrak{M}(T,n,i) \quad \ldots\ldots\ldots\ldots\ldots \quad 1$$

$$\mathfrak{M}(T) = \bigcap_{i=1}^{m} \mathfrak{M}(T,i). \quad \ldots\ldots\ldots\ldots\ldots \quad 2$$

Proof

The second statement is trivial.

The proof of 1, using the local finiteness of $\{N_c\}$, is exactly the same as the proof of Step B in Abraham 1.

PART 3

Proposition C

For all $T,n,c,i;l,m=1,\dots,k_c$ & $r\geqslant 1$, $\mathfrak{M}(T,n,c,i;l,m)$ is open in $\mathfrak{b}(T)$.

Proof

$N_c^{s,l}(n)_i \times N_c^{u,m}(n)_i$ is compact. Now the image of $N_c^{s,l}(n)_i \times N_c^{u,m}(n)_i$ by $I_{l,m}$ is disjoint from a nbd. of the boundary of $\Delta(M_i)$ in $M \times M$. Thus, using Proposition A, we may regard $\Delta(M_i)$ as effectively closed & the result follows using the openness of transversal intersection.

PART 4

We now prove a density result. We will prove:

Proposition D

$$\mathfrak{M}(T,\infty,c,i;l,m) \overset{\text{def}}{=} \bigcap_{n\in Z^+} \mathfrak{M}(T,n,c,i;l,m) \quad \text{is dense}$$

in N_c.

Having shown this, we will then complete the proof of Theorem 31.

First, using Remark 2 on page 140, we may suppose that the 'V_l' nbds in Proposition A, are chosen to satisfy the additional conditions refered to in the remark.

To prepare for the proof of Proposition D, we fix c,i,l,m & use the following simpler notation:

$N=N_c$,

$N^s=N_c^{s,l}$, $N^u=N_c^{u,m}$, $I_Y^s=I_{c,Y}^{s,l}$, $I_Y^u=I_{c,Y}^{u,m}$

$p_Y=p_Y^l$, $q_Y=p_Y^m$,

$V_p=V_c^l$, $V_q=V_c^m$

$$\mathfrak{M}(p,q)=\bigcap_{n\in Z^+}\mathfrak{M}(T,n,c,i;l,m)$$

Thus $\mathfrak{M}(p,q)$ is the set of all vector fields $Y\in N$, s.t. $I^s_Y \times I^u_Y \pitchfork^G \Delta(M_i)$.

We wish to show that $\mathfrak{M}(p,q)$ is dense. Now we have that:

$$I^s_Y : N^s \longrightarrow M \ \&$$

$$I^s_Y(e)=F^Y(G(Y,e)).$$

Now, by choice of V_p, G depends only on $Y|V_p$. Thus, if $Z|V_p=Y|V_p$, $G(Z,e)=G(Y,e)$, $\forall e\in N^s$.

Fix $Y\in N$. We define $\mathfrak{a}\subset N$ to be the set of all $Z\in N$, s.t. $Z|V_p\cup V_q=Y|V_p\cup V_q$. Then $\mathfrak{a}$ is the intersection of a closed hyperplane of $C^r_G(TM)$ with N & hence is a C^∞ Banach Manifold.

By virtue of the above remark, for $Z\in\mathfrak{a}$, $G(Z,e)$ is independent of the choice of $Z\in\mathfrak{a}$. Since the flow $F^Z_t(x)$ is C^r in all three of its variables (this is the parametrised flow theorem, Abraham 1) we have that the map:

$$\mathfrak{a} \longrightarrow C^r(N^s \times N^u, M \times M), \text{given by}$$
$$Z \longmapsto I^s_Z \times I^u_Z, \text{is a } C^r \text{ representation (see}$$

Abraham 1 for definition).

We now prove a Lemma, which corresponds to the 'Main Lemma' of step D in Abraham 1. Our proof is a straightforward generalisation of the proof given in Abraham 1.

<u>Lemma A</u>

The evaluation map, $ev:\mathfrak{a}\times N^s_{\bar{I}}\times N^u_{\bar{I}}\longrightarrow M_{\bar{I}}\times M_{\bar{I}}$, given by $ev(Z,p_1,p_2)=(I^s_Z(p_1),I^u_Z(p_2))$ is transversal to $\Delta(M_{\bar{I}})$, i.e. $ev\pitchfork\Delta(M_{\bar{I}})$. By $M_{\bar{I}}$ we mean M_{G_z} for some $z\in M_i$. We use this interpretation of $\bar{I}$ in the sequel.

Proof

For $Z\in\mathfrak{A}$, we may identify $T_Z\mathfrak{A}$ with the closed subspace F of $C^r_G(TM)$ defined by:

$$F=\{\xi\in C^r_G(TM):\xi|V_p\cup V_q=0\}.$$

With this definition, we see that:

$$T_{(Z,p_1,p_2)}ev(\xi,\dot{p}_1,\dot{p}_2)=\frac{d}{d\zeta}\{ev(Z+\zeta\xi,p_1,p_2)\}_{\zeta=0}+(T_{2(Z,p_1,p_2)}ev)\dot{p}_1$$
$$+(T_{3(Z,p_1,p_2)}ev)\dot{p}_2 \quad \ldots\ldots\ldots \quad A.$$

where $(\xi,\dot{p}_1,\dot{p}_2)\in F\times T_{p_1}N^s_I\times T_{p_2}N^u_I$.

Now suppose that $ev(Z,p_1,p_2)\in\Delta(M_I)$, say $I^s_Z(p_1)=I^u_Z(p_2)=x$, where $x\in M_I$. We must show:

$$T_xM_I\times T_xM_I=T_{(x,x)}\Delta(M_I)+(T_{(Z,p_1,p_2)}ev)(T_Z\mathfrak{A}\times T_{p_1}N^s_I\times T_{p_2}N^u_I),$$

in fact we prove more:

$$T_xM_I\times T_xM_I=T_{(x,x)}\Delta(M_I)+T_{(Z,p_1,p_2)}ev(T_Z\mathfrak{A}\times\{0\}\times\{0\}).$$

We consider first the case $p\neq q$ & show that if $x\notin\bar{V}_p\cup G(q_Z)$, then the first term of A, as ξ varies over F, spans $T_xM_I\times\{0\}$, if $x\notin\bar{V}_q\cup G(p_Z)$, then this term spans $\{0\}\times T_xM_I$.

Thus it suffices to prove:

I. If $x\notin\bar{V}_p\cup G(q_Z)$, then for every $\dot{x}\in T_xM_I$, there exists $\xi\in F$, s.t.:

$$\frac{d}{d\zeta}\{I^s_{Z+\zeta\xi}(p_1)\}_{\zeta=0}=\dot{x}.$$

II. If $x\notin\bar{V}_q\cup G(p_Z)$, then for every $\dot{x}\in T_xM_I$, there exists $\xi\in F$, s.t.:

$$\frac{d}{d\zeta}\{I^u_{Z+\zeta\xi}(p_2)\}_{\zeta=0}=\dot{x}.$$

We prove II, the proof of I is identical.

Now $I^u_Z(p_2)=F^Z(G(Y,p_2))=F^Z(t,y)$, where y is independent of $Z\in\mathcal{Q}$. Since $x\notin\nabla_q\cup G(p_Z)$ & since we have assumed $\nabla_p\cap\nabla_q=\emptyset$, we may find real numbers u & v, with $u<v<t$, such that:

$$F^Z(s,y)\notin\nabla_p\cup\nabla_q,$$

for all $u\leq s\leq v$.

We recall the following perturbation Lemma, adapted from Abraham 1:

<u>Lemma (Abraham)</u>

Let M be a compact G manifold, $r\geq 2$, $\xi^o,\beta\in C^r_G(TM)$, F^o be the flow of ξ^o, & F^ζ (for $\zeta\in R$) be the flow of the vector field $\xi^\zeta=\xi^o+\zeta\beta$.

Then, for $x\in M$ & $t\in R$:

$$\frac{d}{d\zeta}\left\{F^\zeta_t(x)\right\}_{\zeta=0}=\int_0^t TF^o_s.\beta.F^o_{-s+t}(x)ds.$$

Thus we have, for $\xi\in F$, that:

$$\frac{d}{d\zeta}\left\{I^u_{Z+\zeta\xi}(p_2)\right\}_{\zeta=0}=\frac{d}{d\zeta}\left\{F^{Z+\zeta\xi}_t(y)\right\}_{\zeta=0}$$

$$=\int_0^t TF^Z_s.\xi.F^Z_{-s+t}(y)ds \quad \ldots\ldots\ldots \quad C.$$

Now let $g:R\longrightarrow R$ be a C^∞ function s.t.:

1. $g(s)=0$, for $s<u$ or $v<s$.
2. $\int_0^t g(s)ds=1$.

Choose $\dot{x}\in T_xM_I$. Define $\xi|(F^Z_{-s+t}(y))$ for $0\leq s\leq t$ by:

$$\xi(F^Z_{-s+t}(y))=g(s)(TF^Z_{-s})\dot{x}$$

Since F^Z is an equivariant flow & $\dot{x}$ is G_x invariant we may extend ξ to a C^r equivariant vector field, with

$\xi|V_p \cup V_q = 0$, i.e. $\xi \varepsilon F$. But

$$\int_0^t TF_s^Z.\xi.F_{-s+t}^Z(y)ds = \int_0^t g(s)\dot{x}ds = \dot{x}.$$

This equation, together with C., completes the proof of II.

If p=q, the proof is almost identical, with condition 5 of Proposition A, replacing $\bar{V}_p \cap \bar{V}_q = \emptyset$, following I or II according as t is positive or negative.

We now come to the proof of Proposition D:

Proof of Proposition D

If $Z \varepsilon N_c$ & U is an open nbd. of Z in N_c, let $\mathfrak{a}_Z$ be the hyperplane through Z defined above.

Then, provided $\dim M + 1 \leqslant r$, $\mathfrak{a}_Z \cap \mathfrak{M}(T,\infty,c,i;l,m)$ is dense in $\mathfrak{a}_Z \cap N_c$, by the transversality density theorem (Abraham 1) This follows since $\mathrm{codim}(\Delta(M_I))$ in $M_I \times M_I = \dim M_I \leqslant \dim M$, & $\dim(N_I^s \times N_I^u) \leqslant 2\dim M$, & $ev \bar{\pitchfork} \Delta(M_I)$ by lemma A (Here we are using the fact that we may frame our G-transversality definitions in terms of M_i or M_I -see section 14). Thus:

$$U \cap \mathfrak{M}(T,\infty,c,i;l,m) \neq \emptyset$$

This is also true for $r \geqslant 1$, since we may arbitrarily approximate Z by an element of differentiability class $C^{\dim X+1}$ (in fact C^∞) in the C^r topology.

Proof of Theorem 32

In part 1, we constructed a locally finite covering of $\mathfrak{b}(T)$ by open sets N_c & in Part 2 we obtained:

$$\mathfrak{M}(T,i) = \bigcap_{n \varepsilon Z^+} \mathfrak{M}(T,n,i)$$

$$\mathfrak{M}(T,n,i) = \bigcup_{c\in Z^+} \bigcap_{l,m=1}^{k_c} \mathfrak{M}(T,n,c,i;l,m)$$

Proposition C gives us that $\mathfrak{M}(T,n,c,i;l,m)$ is open & thus also the finite intersection over $i,j=1,..,k_c$ & the union over $c\in Z^+$ is open. Thus $\mathfrak{M}(T,n,i)$ is open.

As $\mathfrak{M}(T,\infty,c,i;l,m)$ is dense in N_c, so is $\mathfrak{M}(T,n,c,i;l,m)$ & also $\mathfrak{M}(T,n,i)$. Thus $\mathfrak{M}(T,n,i)$ is open & dense in $\mathfrak{M}(T,i)$ & so in $C^r_G(TM)$ as well. Thus:

$$\mathfrak{N}^{3*}_G(M;r) = \bigcap_{i=1}^{m} \bigcap_{T,n\in Z^+}^{\infty} \mathfrak{M}(T,n,i)$$

is residual.

As an easy corollary of this result we have:

Theorem 33

$\mathfrak{A}^{3*}_G(M;r)$ is residual in $\mathfrak{Diff}^r_G(M)$.

Up till now we have assumed that M was compact, in fact Theorem 31 & Theorem 32 are both true for M non-compact, with the C^r Whitney topology on $C^r_G(TM)$ or $\mathfrak{C}^r_G(M,M)$-see Peixoto 1 for details.

We will not prove this result in detail, but merely note that in our proofs of genericity all we really used, as regards G, was the local finiteness of the G orbit type decomposition of M. Noting this fact the proof for non-compact M goes through formally as in Peixoto 1.

We note in particular that, for non-compact $M, C^r_G(TM)$ still has the Baire property, with Whitney C^r topology.

Part 3: Miscellaneous Results on Equivariant Vector Fields

1. Introduction

In Part 3,we formulate definitions for equivariant Morse-Smale vector fields ('G-Morse-Smale systems') & for equivariant Anosov diffeomorphisms ('G-Anosov maps').

The natural questions to ask of these two classes of equivariant vector fields are:Q1. Do they exist? ; Q2. Are they,in some sense,structurally stable?.The answer to Q1 is yes,in both cases;to Q2 is yes for G-Anosov maps & unknown for G-Morse-Smale systems.

The study of the above systems illustrates some of the difficulties that arise in the study of equivariant vector fields.These difficulties center round the non-compactness of elements of the G-orbit type decomposition of M.Thus we have the problem,mentioned earlier,of G-transversality of embedded compact G manifolds being a non-open condition.

Further,there is a related problem in defining,in general,what one means by a 'hyperbolic set' for equivariant systems.

In Part 3,we only take a brief look at the above problems indicating difficulties & possible lines of approach.

2. 'G-Morse-Smale systems'

Recalling the definition of the 'Ω-set' from Smale 1, we have:

Definition 38*

Let $f \in \mathfrak{Diff}^r_G(M), r \geq 1$, we say f is a 'G*-Morse-Smale diffeomorphism' if the following three conditions hold:

1. $\Omega(f)$, mod G, is finite.
2. $\Omega(f)$ consists of families of closed orbits & fixed points.
3. f is 3-generic.

Similarly for vector fields.

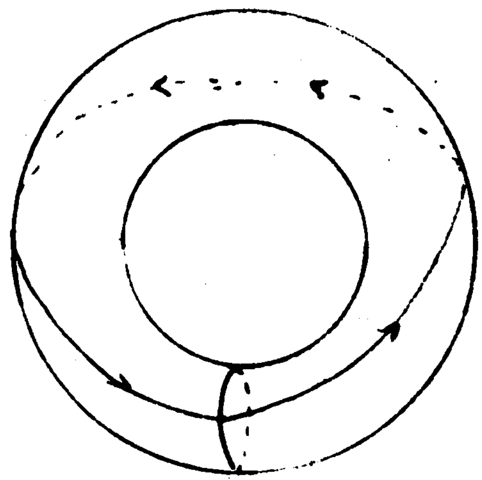
We note here that, unlike the G=id case, $\Omega(f)$, mod G, finite, does not imply $\Omega(f)$=Per(f): The set of periodic points of f.

Now, if f is G*-Morse-Smale, f is 2*-generic &, in general, every G-manifold does not admit a G*-Morse-Smale system:

Example 1

We consider the group T^2 under its own action.

Then we note that if $X \in C^\infty_{T^2}(T^2)$, then the flow of X either consists of an S^1 family of closed orbits, -in which case X is 2-generic but not 2*-generic, since $\operatorname{rank}(N(T^2_x)/T^2_x) > 1, x \in T^2$, or



consists of an irrational flow, when X is 2*-generic but not

G*-Morse-Smale.

Thus we consider instead:

Definition 38

If f is an equivariant diffeomorphism, we say f is a 'G-Morse-Smale diffeomorphism' if f satisfies conditions 1. & 2. of Definition 38*, but we relax condition 3. to require only that f is 1. & 2. generic, with the stable & unstable manifolds still meeting G transversally.

Similarly for vector fields.

We will show that every compact G-manifold admits a G-Morse-Smale system. We note, however, that, in contrast to G*-Morse-Smale systems, G-Morse-Smale systems will not, in general, form an open set, nor even have a non-void interior, as we can generally perturb 2-generic closed orbits into families of non-compact recurrent orbits. We do, however, by Theorem 22, locally preserve the normal hyperbolicity, so we can reasonably formulate:

Definition 38$^!$

A $G^!$-Morse-Smale diffeomorphism f, satisfies:

1. $\Omega(f)$, mod G, finite.
2. Each element of $\Omega(f)$ is an f-invariant G normally hyperbolic set (perhaps for some power of f)
3. The stable & unstable manifolds of elements of $\Omega(f)$ are G-transversal.

Similarly for vector fields.

With definition 38$^!$ we might hope to prove some openness properties.

We now come to the main theorem of this section.

Theorem 34

Every compact G-manifold admits a G-Morse-Smale vectorfield.

Proof

First, by a theorem of Wasserman 1, there exists a G Morse map for M, with critical locus a union of non-degenerate critical orbits, let us denote such a map by f.

If we consider grad(f), w.r.t. an equivariant Riemannian metric, we get an equivariant 1 (& 2) generic vector field X_f=grad(f) on M, with no closed orbits. We have to show that we may perturb X_f to X' s.t. the stable & unstable manifolds of X' are G-transversal and no new fixed sets are introduced.

This is easy, using the methods of Theorem 32. We take G-invariant nbds $V_1, \dots, V_k$ of the singular sets $G(p_1), \dots, G(p_k)$ of X_f as in Proposition A. Then we consider the set of equivariant vector fields $\mathfrak{A}_{X_f} = \{X \in C^r_G(TM) : X|(V_1 \cup \dots \cup V_k) = X_f|(V_1 \cup \dots \cup V_k)\}$, & show we may approximate X_f by X' in $\mathfrak{A}_{X_f}$ s.t. X' has G-transversal stable & unstable manifolds of critical elements (We may assume no new singular sets or closed orbits are introduced, since X_f is transversal to the level surfaces of f, outside of $\bigcup V_i$). We omit details.

We note that we may choose an equivariant Riemannian metric for M, s.t. X'=grad(f), w.r.t. this new metric, see Smale 2.

Now, since $\Omega(X)=\mathrm{Per}(X)$, for a G-Morse-Smale system, it follows that $M=\bigcup W^u(G(p_i))=\bigcup W^s(G(p_i))$, where p_i are the critical elements of X. Whilst it does not appear difficult to show that $W^u(G(p_i))_j=W^u(G(p_i))\cap M_j$ is an embedded submanifold of M, the question of whether or not $W^u(G(p_i))$ is an embedded submanifold is more difficult (& doubtful without more conditions). Similarly, we do not get the nice ordering of the elements of $\Omega(X)$, for a G-Morse-Smale system that occurs for the G=id case (Smale 3). However, we will end this section with some very tentative definitions about structural stability.

First, let S(G,M) denote the set of G-Morse-Smale vector fields on M. We may ask:

Q1. Is S(G,M) open in $C^r_G(TM)$?-at least in the sense of Definition 38'.

Q2. Are elements of S(G,M) structurally stable? Does every G manifold admit a structurally stable element of S(G,M)?

The disadvantage of G-transversality is that it is not, in general, an open relation for compact manifolds. This indicates that the answer to Q1 & Q2 must, in general, be no. However we introduce a new topology on $C^r_G(TM)$ in which Q1 & Q2 become reasonable possibilities. Basically we wish to exercise tighter & tighter control on X as we approach the boundary of a G-orbit type component of M

First we recall the definition of the Whitney C^r topology, $r\geqslant 1$ (Peixoto 1).

If M is non-compact let $K_1\subset\ldots K_i\subset K_{i+1}\ldots\subset M$ be a decomposition of M into an expanding sequence of compact sets, each K_i having non-zero interior $\mathring{K}_i \subset K_{i+1}$. Let $X \in C^r(TM)$ & let $d(x)>0$ be a C^0-function defined on M. Let

$$d_i = \inf(d(x)), x \in K_i - \mathring{K}_{i-1}, K_0 = \emptyset.$$

We set

$$A(X,d(x)) = \bigcap_{i=1}^{\infty} \{Y \in d(X,Y;K_i - \mathring{K}_{i-1}) < d_i\},$$

where $d(X,Y;K_i-\mathring{K}_{i-1})$ stands for the usual C^r distance between X & Y. When $d(x)$ varies in the set of C^0 functions on M, the sets $A(X,d(x))$ form a basis in X for a system of nbds of a topology in $C^r(TM)$, which does not depend on the decomposition K_i of the space M. If M is compact, then we get the usual C^r topology on $C^r(TM)$.

Now the crucial point to note is that, in the decomposition of M, $M = M_1 \cup .. \cup M_i \cup .. \cup M_m$, the M_i are in general non-compact.

Let $X \in C^r_G(TM)$. On each M_i we define, as above, a Whitney C^r topology-recall that $X|M_i \in C^r_G(TM_i)$, using the previously defined C^r norm for $C^r_G(TM)$.

Finally we define:

$$A(X,\underline{d}(x)) = \bigcap_{i=1}^{m} \bigcap_{j=1}^{\infty} \{ Y : d(X_i, Y_i; K^i_j - \mathring{K}^i_{j-1}) < d^i_j\},$$

where $X_i = X|M_i$, $\{K^i_j\}_{j \in Z^+}$ is the decomposition of M_i & $\underline{d} = (d_1, \dots, d_m) : M_1 \times .. \times M_m \longrightarrow R^+$.

Then, as above, we have a base of nbds for X which defines what we term a 'C^r G Whitney topology' on M.

Since $C^r_G(TM_i)$ is Baire, so is $C^r_G(TM)$ with the above topology; it is not a Banach space, or even metrisable.

Remarks:

1. We recall the proof of Theorem 32.With the above topology on $C^r_G(TM)$ G-transversality of '$N^s(n)$' obviously becomes an open condition,as in the G=id case.

2. We may similarly define a Whitney G topology on $C^r_G(E)$,where $E\in GFB(M)$ & in particular on $\mathfrak{Diff}^r_G(M)$.

3. Suppose G & M are connected then (see Borel 1),if $M=M_1\cup..\cup M_m$ is the orbit decomposition of M,one of the M_i is open & dense in M.This M_i dominates the Whitney G C^r topology:For suppose $d_i:M_i \longrightarrow R^+$ is s.t. $d(x)\longrightarrow 0$ as $x\longrightarrow bndry(M_i)$,then,if we consider the nbd. $A(X,\underline{d}(x))$ of X,we will have for $Y\in A(X,\underline{d}(x))$,Y=X on $M-M_i$

For the remainder of this section we will restrict our attention to equivariant diffeomorphisms.

Definition 39

Let $f\in\mathfrak{Diff}^r_G(M)$.

We say f is G-structurally stable if there exists a nbd. U_f of f,in the Whitney G C^r topology,s.t.,$\forall g\in U_f$, there exists:

1. An equivariant homeomorphism h of M.
2. A continuous map $Q:M\longrightarrow V\subset G$,where V is a small nbd. of the identity in G^o s.t.:

   a) $Q(fx)h(fx)=gh(x),\forall x\in M$.

   b) The map $Q.h:M\longrightarrow M$,is an equivariant homeomorphism.

Remark:

We have to introduce Q to take account of fixed points of f perturbing into periodic or non-compact orbits.

We say f is strong G structurally stable if we can find a nbd. U_f of f, s.t. Q=e for all $g \in U_f$.

Definition 40

We say f is weak G structurally stable if there exists a nbd. U_f of f in the Whitney C^r G topology s.t. if $g \in U_f$, & $\bar{f}$ & $\bar{g}$ denote the maps induced by f & g on the orbit space M/G, then $\bar{f}$ & $\bar{g}$ are C^0 conjugate on M/G.

Remark:

Clearly G structural stability implies weak G structural stability, with conjugating map $\bar{h}$.

We end the section with the following problems:

Problems

1. Are $G^!$ Morse-Smale systems weak G-structurally stable? (in the Whitney G topology)
2. Are G*-Morse Smale systems (strong?) G structurally stable (in the Whitney $\mathcal{G}$ C^r topology)?.

We have not investigated any of the above problems.

3. 'G-Anosov Maps'

In this section we will restrict our attention to equivariant diffeomorphisms.

Thus suppose f is an equivariant diffeomorphism of M, then it is easy to check that f defines an isomorphism $f_*:C^0_G(TM)\longrightarrow C^0_G(TM)$ by:

$$f_*(X(x))=Tf(X(f^{-1}(x))),\text{for } x\in M, X\in C^0_G(TM).$$

Definition 41

If $f_*:C^0_G(TM)\longrightarrow C^0_G(TM)$ is hyperbolic (i.e. spectrum disjoint from unit circle) we say that f is a 'G-Anosov map'.

Example 1

G:Finite.

Let us consider the $Z_2 \times Z_2$ action on T^2 induced from the "$Z_2 \times Z_2$" action on R^2, given by:

$$(x,y)\longrightarrow(x-p\tfrac{1}{2},y-q\tfrac{1}{2}),\text{where}$$

p & $q\in\{0,1\}$.

Now it is easy to check that the map induced on T^2 by:

$$\begin{pmatrix}1 & 4\\ 2 & 9\end{pmatrix}$$ is $Z_2 \times Z_2$ invariant & is an Anosov Diffeomorphism. It follows that it is a $Z_2 \times Z_2$ Anosov map. We note that it is structurally stable.

Example 2

Consider $T^2 \times S^2$, & $G=id \times SO(3)$, acting on $T^2 \times S^2$. We define $A:T^2 \times S^2 \longrightarrow T^2 \times S^2$, by $A(x,y)=(Tx,y)$, where T is the Thom map on the torus, induced from:

$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

We consider the map $A_*: C^0_G(T(T^2 \times S^2)) \longrightarrow C^0_G(T(T^2 \times S^2))$.

One may easily check that $\bar{A}_*$ is hyperbolic (essentially A is an SO(3) parametrised family of Anosov diffeomorphisms on T^2, all equal to the Thom map).

Further A is structurally stable (since our choice of G stops perturbation out of the fibers of $T^2 \times S^2 \longrightarrow S^2$).

Problem

Find less trivial examples of G-Anosov maps.

Our main theorem is the following:

Theorem 35

If f is a G-Anosov map, then f is (equivariantly) structurally stable.

Moreover if u is the equivariant homeomorphism satisfying $g = u^{-1}.f.u$, then u depends continuously on f,g if the mapping u is considered in the C^0 topology & f,g in the C^1 topology. (Here we are using the usual C^r topologies for M, & not the Whitney G topology)

Also we have:

Theorem 36

The G-Anosov maps form an open set in the space of equivariant diffeomorphisms.

The above Theorems are generalisations of the Theorems of Anosov. For proof for the G=id case & discussion of Anosov diffeomorphisms see Moser 1.

Proof of Theorem 35

Here we follow very closely Moser's proof for G=id, in Moser 1:Essentially we choose an equivariant Riemannian metric for M & check that in Moser's proof we have equivariant maps defined in Lemmas 1 & 2.Lemma 3 follows as in Moser 1 & the rest of the proof follows formally as in Moser 1.

Proof of Theorem 36

Again a 'G-version' of Moser 1.We omit details.

Problems

1. Give a chacterisation of G-Anosov maps,other than that of Definition 41.

This would seem to relate closely to ideas about hyperbolic sets for equivariant maps.

2. Does $\Omega(M/G)$ for $\bar{f}$,f G-Anosov,=M/G.etc.

Appendix 1

In this Appendix we state & prove some of the results about Lie group actions that we have used in the text. With the possible exception of Theorem 1 all these results are well known & can be found else-where, they are collected here for reference. I have not found an explicit proof or statement of Theorem 1.

Lemma 1

Let S_x be a slice at $x\varepsilon G(x), x\varepsilon M$.

Define $S_1=\{z\varepsilon S_x : G_z=G_x\}$.

Then in an appropriate coordinate system on S_x, G_x acts as an orthogonal group of transformations & $S_x=S_1 \times S_2$, where S_2 is a G_x invariant orthogonal complement to S_1, w.r.t. some G_x-invariant metric on S_x.

Proof

This is just a statement about the G-vector bundle associated to S_x, see section 9, the proof is easy.

If M is compact we know $M=M_1\cup\ldots\cup M_m$, where M_i is of fixed orbit type.

Also it is clear that, since M_i is a submanifold of M,

$$M_i=\bigcup_{j\varepsilon J_i} M_i^j,$$

where the M_i^j are the connected submanifold componenets of M_i. Suppose $x\varepsilon M_i^j$, let $M_i^{j,x}=\{z\varepsilon M_i^j : G_z=G_x\}$.

Proposition 1

$M_i^{j,x}\subset M_i^j$ as a submanifold.

Proof

This is obvious, by considering the fixed point set of the induced G_x action on M_i.

Next we show that each M_i is a finite union of connected submanifold components M_i^j, i.e. J_i is finite. We first prove a Lemma:

Lemma 2

Let G be a compact Lie group of transformations acting differentiably on R^n as a group of orthogonal transformations.

Suppose $R^n = K_1 \cup \ldots \cup K_N$ is the decomposition of R^n into orbit types.

Let $K_i = \bigcup_{j \in J_i} K_i^j$ be the decomposition of K_i into connected submanifold components.

Then J_i is finite for $i=1,..,N$.

Proof

We first make the obsevation that the orbit decomposition of R^n is finite. This is so since G acts on $S^{n-1} \subset R^n$ & if $\{K_i'\}$ denotes the orbit decomposition of S^{n-1} (finite, since S^{n-1} is compact) then $K_i' = S^{n-1} \cap K_i$, since G acts linearly on R^n.

We prove the lemma by induction on n.

1. n=1, trivial.
2. We suppose the lemma true for m<n.

Let us suppose that for some i, J_i is not finite. Then we may construct a sequence $\{x_r\}_{r \in J_i}$, s.t. $x_r \in K_i^r \cap S^{n-1}$ &

$x_r \longrightarrow x \in S^{n-1}$ & no two of the x_r belong to the same component of K_i. Thus $x \in K_s$ for some s.

Now since S^{n-1} is G-invariant we have a slice S_x at x in S^{n-1} & G_x acts orthogonally on S_x-lemma 1.

Let G^0 be the identity component of G. Then $\exists R \in Z^+$, s.t. $x_r \in G^0(S_x), \forall r \geq R$.

Suppose $x_r \in u_r(S_x), u_r \in G^0, r \geq R, x_r \in K_i^r$. Let $\{u_r^t\}_{t \in [0,1]}$ be a path from e to u_r^{-1} in G^0. Then $u_r^t(x_r) \in K_i^r$ & consequently $u_r^1(x_r) \in K_i^r$. Therefore we may suppose, w.l.o.g. that $x_r \in S_x, r \geq R$.

Considering the action of G_x on S_x we may write:

$S_x = K_1'' \cup \ldots \cup K_M''$, as the G_x orbit decomposition of S_x. But it is easy to see that a point $z \in S_x$ is of type j in S_x iff it is of type j in R^n & $K_j'' = K_j \cap S_x$ (some K_j'' may be empty). This is so since $(G_x)_y = G_y, y \in G_x$.

As a consequence of this, we note that x_r & x_s belong to different components of K_i'' if they belong to different components of K_i.

Contradiction by our inductive hypothesis, since $\dim(S_x) < n$.

Theorem 1

If G is a compact Lie group of transformations acting differentiably on a compact manifold M & if $M = M_1 \cup \ldots \cup M_m$, is the decomposition of M into orbit types, the each M_j may be written as a finite union of connected submanifold components of M_j.

Proof

We may certainly write $M_j = \bigcup_{i \in I_j} M_j^i, M_j^i$: connected submanifold component of M_j.

We suppose I_j is not finite for some j. Then we may construct a sequence $x_r \in M_j^r$, s.t. $x_r \longrightarrow x$ & no two points of the sequence belong to the same component of M_j.

As in Lemma 2 we may assume $x_r \in S_x$, $r \geq R \in Z^+$, where S_x is a slice at x. Lemma 2 gives us that we have only a finite number of connected components of type j in S_x. Contradiction, therefore I_j is finite.

Appendix 2

This appendix constitutes a brief survey, with proofs, of the theory of vector bundle approximations. In particular, we prove results about G vector bundle approximations, which we have used in several places in the main part of the text.

Notation

M: Compact C^r manifold, $\infty \geq r \geq 1$.

$VB_n^s(M)$ denotes the category of C^s vector bundles over M, $0 \leq s \leq r$, with fiber dimension n. We let $VB^s(M)$ denote the category of all C^s vector bundles over M & $VB(M)$ denote the category of C^∞ vector bundles over M (Here M is, of course C^∞).

We give similarly for $FB_n^s(M)$, $FB^s(M)$ & $FB(M)$.

If M is a compact G manifold, we give the appropriate meanings to $GVB_n^s(M)$ etc.

We will follow Husemöllier [1] in the sequel, in particular pages 100 & 101. All unproved assertions in the sequel, without reference, will be found explicitly or implicitly there.

Let $E \in VB_n^s(M)$, then we have a classifying map $f: M \longrightarrow G_n(R^m)$, s.t. if γ_n^m denotes the classifying bundle, then:

$$E \text{ is } C^s \text{ VB isomorphic to } f^*(\gamma_n^m).$$

Lemma 1

If $f\in\mathfrak{C}^s(M,N)$, then $\exists$ a nbd. U_f of f in $\mathfrak{C}^s(M,N)$ s.t. all $g\in U_f$ are C^s homotopic to f.

Here M & N are of class at least $C^{\max(s,1)}$, & s may =0. $\mathfrak{C}^s(M,N)$ has the usual C^s topology.

Proof

We take C^{∞} structure on N, compatible with the original structure on N. We then proceed exactly as in Lemma 9, using a Riemannian metric on N. We omit details.

Using well known approximation theory we have:

Corollary 1.1

If $f:M\longrightarrow G_k(R^m)$ is C^s & M is C^r, $r>s$, we may approximate f by a C^r map f', which is C^s homotopic to f.

Definition 1

If $E\in VB_n^s(M)$ & M is C^r, then we say that E' is a C^r representation of E, if E' is C^s VB isomorphic to E & $E'\in VB_n^r(M)$.

Theorem 1

If $E\in VB_n^s(M)$ & M is C^r, then E has a C^r representation.

Proof:

Just corollary 1.1.

Suppose $E\in VB_n^r(M)$ & $E_1\in VB_{m_1}^s(M)$ is a C^p subbundle of E. Here $0\leq s\leq r$, $0\leq p\leq s$.

Definition 2

With the above notation, we say E_1' is a C^r approximation to E_1 (in E) if:

1. E_1' is a C^r subbundle of E.
2. E_1' is C^0 close to E_1.
3. E_1' is C^s VB isomorphic to E_1.

N.B. By 'C^0 close' we mean here that the unit sphere bundle of E_1' is close to the unit sphere bundle of E_1 in E.

Theorem 2

With the notation of definition 2, E_1 has a C^r approximation E_1' in E,

Proof

Let E_2 be an orthogonal complement for E_1 in E, $E_2 \varepsilon VB^s_{m_2}(M)$.

Let $f_i : M \longrightarrow G_{m_i}(R^{t_i})$ be the classifying map for E_i, $i=1,2$. Then we have a classifying map map for E, f, given by the commutativity of:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "(f_1,f_2)"'] & G_n(R^{t_1+t_2}) \\ & G_{m_1}(R^{t_1}) \times G_{m_2}(R^{t_2}) \arrow[u, "d"'] \end{tikzcd}$$

We have the following commutative diagram of C^0 VB isomorphisms & subbundle inclusions:

$$\begin{tikzcd} E \arrow[r, "\cong"] & f^*(\gamma_n^{t_1+t_2}) \\ E_1 \arrow[u, "\subset"'] \arrow[r, "\cong"] & f_1^*(\gamma_{m_1}^{t_1}) \arrow[u, "\subset"'] \end{tikzcd}$$

Let us now take C^r approximations f_1', f_2' to f_1, f_2 respectively, giving a C^r map $f' = (f_1', f_2')$. C^r close to f.

Then we get by Lemma 1 a new diagram of C^0 VB isomorphisms & subbundle inclusions:

$$\begin{tikzcd} E \arrow[r] & f^*(\gamma_n^{t_1+t_2}) \arrow[r, "\simeq"] & f'^*(\gamma_n^{t_1+t_2}) \\ E_1 \arrow[r] \arrow[u, phantom, "\cup"] & f_1^*(\gamma_{m_1}^{t_1}) \arrow[r, "\simeq"] \arrow[u, phantom, "\cup"] & f_1'^*(\gamma_{m_1}^{t_1}) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Hence we have:

$$\begin{tikzcd} E & f'^*(\gamma_n^{t_1+t_2}) \arrow[l, "A"'] \\ E_1 \arrow[u, phantom, "\cup"] & f_1'^*(\gamma_{m_1}^{t_1}). \arrow[l, "B"'] \arrow[u, phantom, "\cup"] \end{tikzcd}$$

But all the bundles on the top row are C^r. Thus we may approximate A by a C^r VB isomorphism & hence induce a subbundle map $B': f_1'^*(\gamma_{m_1}^{t_1}) \longrightarrow E$, whose image is the C^r approximation to E_1 in E required.

As an easy corollary we have:

<u>Corollary 2.1</u>

If $E \in VB_n^r(M)$ & $E = \bigoplus_{i=1}^{N} E_i$, where $E_i \in VB_{n_i}^{s_i}(M)$, $0 \leq s_i \leq r$. Then we may take C^r approximations E_i' to E_i s.t.:

$$E = \bigoplus_{i=1}^{N} E_i'.$$

We now restrict attention to G vector bundles. If $E \in GVB^r(M)$ & E_1 is a C^s G subbundle of E, we have the following generalisation of Definition 2:

Definition 3

With the above notation, we say E_1' is a C^r G approximation to E_1 (in E) if:

1. E_1' is a C^r G vector subbundle of E.
2. E_1' is C^o close to E_1.
3. E_1' is C^s GVB-isomorphic to E_1.

We now make the following observations:

1. If f & g $\varepsilon\,\mathcal{C}^r_G(M,N)$ & f & g are C^r equivariantly homotopic, then, if $p\varepsilon GVB^r(N)$, we have:

$$f^*p \text{ is } C^r \text{ GVB-isomorphic to } g^*p.$$

This is just a consequence of the standard construction of the isomorphism between f^*p & g^*p. See Atiyah & Segal 1.

2. Lemma 1 has an immediate generalisation to equivariant maps (Essentially Lemma 9).

3. We may regard '$G_n(R^m)$' as a G-Manifold when working with G vector bundles & we get G vector bundles from equivariant classifying maps (See Atiyah 1 or Atiyah & Segal 1).

4. By a theorem of Wasserman 1, if $f:M\longrightarrow N$ is a C^s equivariant map between C^r manifolds it has C^r equivariant approximations.

5. If $A:E\longrightarrow F, E, F\varepsilon GVB^r(M)$ is a C^s GVB isomorphism, then A has a C^r GVB-isomorphism approximation $\bar{A}$.

This follows since A certainly has a C^r VB-isomorphism approximation A^* (covering the identity); define $\bar{A}=Av(A^*)$.

Now 1,...,5 allow us to generalise everything about vector bundle approximations to G vector bundle approximations. In particular we have:

Theorem 3

Let $E \varepsilon GVB^r(M)$ & suppose E_1 is a C^s G vector subbundle of E. Then E_1 has a C^r G vector bundle approximation E_1' to E_1 in E.

REFERENCES for Appendix 2.

Husemollier 1: 'Fiber Bundles', McGraw Hill, 1966.

Atiyah 1: 'K-Theory', Benjamin, 1964.

Atiyah & Segal 1: Notes on Equivariant K-Theory.

Wasserman 1: Equivariant Differential Topology, Topology 8, pages 127-150.

Appendix 3

In this appendix we prove the following theorem for type A closed orbits:

Theorem

Let $s \in C^r_G(TM)$ & let q be a closed orbit of s. Then we may arbitrarily approximate s by s', at the C^r level, s.t. q is a 2-generic closed orbit for s'.

Proof

We follow closely the notation & methods of the second part of section 10.

Let $f_s : N_r \longrightarrow N^*$ denote the generalised Poincaré map for s at q.

Then f_s factors as $f_s = \psi^k$, where $\psi : N_r \longrightarrow N^*$. Let us suppose $G_q = C^m . G_x$ & h is a generator of C^m.

Define $f_h = h^{-1}\psi$. Then $G(x)$ is a fixed set for f_h. Now by the work in section 5, we may arbitrarily C^r approximate f_h by f'_h, s.t. $G(x)$ is a 1-generic fixed set for f'_h.

Using Lemma 9, as in section 10, we find a C^r approximation s' to s, s.t. the generalised Poincare map $f_{s'}$ for s' is given by:

$$f_{s'} = (hf'_h)^k \text{-at least in some nbd. of } G(x) \text{ in}$$

N_r.

Now $(hf'_h)^k = h^k f'^k_h$, since f'_h is equivariant, thus

$$f^P_{s'} = h^{Pk} f'^{Pk}_h = f'^m_h \quad (m = Pk, P \text{ the period of } C^m \text{ w.r.t. } q).$$

But since f'_h is 1-generic on $G(x)$, so is f'^m_h, thus $f^P_{s'}$ is 1-generic on $G(x)$, therefore so is $f_{s'}$ & hence q is a 2-generic closed orbit for s'.

Appendix 4

In this appendix, we briefly consider how we may strengthen Theorem 16, of section 10, so as to require $C(t)\equiv 0$. Having shown that we may suppose $C(t)\equiv 0$, we give a stronger form of Theorem 17, in which we remove the dependence of B on t.

Here we work only with type B closed orbits. The methods for type A closed orbits are similar, but we lose some differentiability.

Let $s\in C^r_G(TM)$ & q be a type B closed orbit of s. First we recall part of Proposition 27, page 9?, which we restate for convenience:

Proposition 27

For type B closed orbits, C is a C^∞ trivial bundle over q.

We recall that C was the othogonal complement of Tq, in $TG(q)|q$.

Now the flow of s, F_t, acts on G(q) as a C^∞ S^1 action, commuting with the G-action on G(q).

Thus we have a C^∞ S^1 x G-action on G(q) & hence on TG(q). We choose a Riemannian metric for G(q), which is S^1 x G invariant & extemd it to a G-invariant Riemannian metric on M. With respect to this metric C is an $S^1 \times G_q$-vector bundle & the orthogonal complement of Tq in $TG(q)|q$.

Then we have a C^∞ trivialisation of C, given by:

$$\begin{tikzcd} S^1 \times C_x \arrow[r, "H"] \arrow[d] & C \arrow[d] \\ S^1 \arrow[r, "h"] & q \end{tikzcd}$$,where

$H(t,\dot{x})=TF_t(\dot{x})$ & x is a fixed point of q.

Here we see, that w.r.t. this trivialisation, $S^1 \times \{\dot{x}\}$ is mapped by H & then by the exponential map onto the closed orbit of s through $\exp_\bullet H(\dot{x})$; this is so since exp is S^1 invariant.

Thus, with this trivialisation of C, we get, in the proof of theorem 16, that $R^1 \times \{0\} \times R^d$ is invariant by the flow & indeed the flow is in the R^1 direction, showing that $C(t)\equiv 0$.

Now given the resulting pseudo-chart representation of s, we wish to remove the dependence of A & B on t.

As in the proof of Theorem 17, we define a new vector field w by:

$$w(t,x,y)=(t,x,y;1,A(t)x,B(t)x)$$

It is clear that the flow of w is of the form:

$$H(t,x,y;s)=(t+s,H_s^1(t)x,y+H_s^2(t)x), \text{where}$$

$H_s^1(t)\in L(R^v,R^v)$ & $H_s^2(t)\in L(R^v,R^d)$.

Now, exactly as in the proof of Theorem 16, we find $A\in L(R^v,R^v)$, s.t. $\exp(2TA)=H_{2T}^1(0)$.

We note that 'A' may be assumed to be equal to the A constructed in Theorem 17, as $H_s^2(t)=G_s(t)$.

We now solve:

$$\exp 2T\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} = \begin{pmatrix} H^1_{2T}(0) & 0 \\ H^2_{2T}(0) & 0 \end{pmatrix}, \text{for } C.$$

Writing this out formally we have:

$$2TC(I+\frac{1}{2!}(2TA)+\frac{1}{3!}(2TA)^2+\dots\frac{1}{n!}(2TA)^{n-1}+\dots)=H^2_{2T}(0).$$

i.e.:

$$2TC=H^2_{2T}(0)\frac{(2TA)}{(\exp(2TA)-I)}.$$

To see that the above gives a solution for C, it is sufficient to note that $\frac{(2TA)}{(\exp(2TA)-I)} \in \mathrm{Iso}(R^v,R^v), \forall A \in L(R^v,R^v)$,

this follows since $\frac{z}{\exp z-1}$ is an analytic function, radius of convergence ∞, see MacRobert 1, page 96.

Thus having found C, we define:

$$P: R \longrightarrow L(R^v \times R^d, R^v \times R^d) \text{ by:}$$

$$P(t) = \exp\begin{pmatrix} tA & 0 \\ tC & 0 \end{pmatrix} \cdot \begin{pmatrix} H^1_t(0) & 0 \\ H^2_t(0) & 0 \end{pmatrix}^{-1},$$

giving a Floquet representation of s as required:

Theorem

$\exists$a GPC $(R \times R^v \times R^d, G^!_q, S)$ for q, s.t. the principal parts of the local representitive of s*, w.r.t. this pseudo-chart, $s^*=(s^*_1,s^*_2,s^*_3)$ have the form:

$s^*_1(t,x,y)=1+Q(t,x,y)$; $s^*_2(t,x,y)=Ax+R_1(t,x,y)$;

$s^*_3(t,x,y)=Bx+R_2(t,x,y)$.

We note that the corresponding Poincare map for q, has linear approximation (to its square):

$$\exp\begin{pmatrix} 2TA & 0 \\ 2BT & 0 \end{pmatrix} = \begin{pmatrix} \exp 2TA & 0 \\ B(\exp 2TA) & I \end{pmatrix}.$$

REFERENCE

MACROBERT 1: Functions of a Complex Variable, MacMillan 1962.

REFERENCES

ABRAHAM R. & ROBBIN J. 1:Transversal Mappings & Flows, Benjamin, 1967.

ABRAHAM R. 2:Foundations of Mechanics, Benjamin, 1967.

ATIYAH 1:K-Theory, Benjamin, 1967.

BOREL 1:Seminar on Transformation Groups, Princeton Univ. Press, 1960.

DIEUDONNÉ 1:Foundations of Modern Analysis, Academic Press, 1960.

KOBAYASHI & NOMIZU 1:Foundations of Differential Geometry, Interscience, 1963.

KOZUL 1:See:A.Gleason, Spaces with a compact Lie group of Transformations, Proc. A.M.S. 1 (1950), 35-43.

LANG 1:Introduction to Differentiable Manifolds, Interscience, 1966.

MONTGOMERRY & ZIPPIN 1:Topological Transformation Groups, Interscience, 1955.

MOSER 1:On a Theorem of Anosov, Journal of Diff. Eqns., 5, 411-440 (1969).

PALAIS 1:Foundations of Global non-linear Analysis, Benjamin, 1968.

PALIS 1:On Morse-Smale Dynamical Systems, Topology 8, pp. 385-405 (1969).

PALIS-SMALE: Structural Stability Theorems,

PEIXOTO 1: On an Approximation Theorem of Kupka & Smale, Journal of Diff. Eqns. 3, 214-227 (1966).

PONTRIAGIN 1: Ordinary Differential Equations, Addison-Wesley, 1962.

PUGH-HIRSCH-SHUB (P-H-S 1): Invariant Manifolds, to appear.

SCOTT 1: Group Theory, Prentice Hall, 1964.

SMALE 1: Differentiable Dynamical Systems, Bull. Am. Math. Soc. Nov. 1967.

2: On Gradient Dynamical Systems, Ann. of Math. (2) 74 (1961) 199-206.

3: Morse Inequalities for a Dynamical System, Bull. Am. Math. Soc. 66 (1960), 43-49.

THOM 1: Quelques propriétes globales des variétés différentiables, Comm. Math. Helv., 28 (1954), 17-86.

WASSERMAN 1: Equivariant Differential Topology, Topology 8 (1969) 127-150.